Preface to the second edition

This second edition contains a good number of additions scattered throughout the text as well as numerous voluntary (and involuntary) changes. Most notably we have added Section 7.5 on the *Potential operator* of a Brownian motion, Chapter 11 *Brownian motion as a random fractal* and Chapter 20 on *Stratonovich's stochastic calculus*. The last addition prompted us to rewrite large portions of our presentation of the stochastic calculus; Chapter 17 on Itô's formula contains now full proofs also of the multivariate and time-dependent cases, and the SDE-chapters (Chapter 19 and 20) put more emphasis on solution strategies for concrete stochastic differential equations. Throughout the book we added further exercises which come with a detailed solution manual on the internet http://www.motapa.de/brownian_motion/. Despite of all of these changes we did not try to make our book into an encyclopedia, but we wanted to keep the easily accessible, introductory character of the exposition, and there are still plenty of topics deliberately left out.

After the publication of the first edition, we got many positive responses from colleagues and students alike, and quite a few led to improvements in the new edition. Thanks are due to our readers for telling us about errors and obscurities in the first edition. Very special thanks go to our student Franziska Kühn (who ploughed through the whole text and knows it inside out) and to Björn Böttcher and Julian Hollender for their valuable contributions. The de Gruyter editorial staff around Ms. Friederike Dittberner did a splendid job for this project and deserves our appreciation.

Without the patience – we tried it hard enough in all those moments when our thoughts were again diffusing in Wiener space – and support of our families this endeavour would not have been possible. Thank you!

Dresden, February 2014

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Preface

Brownian motion is arguably the single most important stochastic process. Historically it was the first stochastic process in continuous time and with a continuous state space, and thus it influenced the study of Gaussian processes, martingales, Markov processes, diffusions and random fractals. Its central position within mathematics is matched by numerous applications in science, engineering and mathematical finance.

The present book grew out of several courses which we taught at the University of Marburg and TU Dresden, and it draws on the lecture notes [172] by one of us. Many students are interested in applications of probability theory and it is important to teach Brownian motion and stochastic calculus at an early stage of the curriculum. Such a course is very likely the first encounter with stochastic processes in continuous time, following directly on an introductory course on rigorous (i. e. measure-theoretic) probability theory. Typically, students would be familiar with the classical limit theorems of probability theory and basic discrete-time martingales, as it is treated, for example, by Jacod & Protter *Probability Essentials* [108], Williams *Probability with Martingales* [232], or in the more voluminous textbooks by Billingsley [15] and Durrett [61].

General textbooks on probability theory cover however, if at all, Brownian motion only briefly. On the other hand, there is a quite substantial gap to more specialized texts on Brownian motion which is not so easy to overcome for the novice. Our aim was to write a book which can be used in the classroom as an introduction to Brownian motion and stochastic calculus, and as a first course in continuous-time and continuous-state Markov processes. We also wanted to have a text which would be both a readily accessible mathematical back-up for contemporary applications (such as mathematical finance) and a foundation to get easy access to advanced monographs, e. g. Karatzas & Shreve [119], Revuz & Yor [189] or Rogers & Williams [195] (for stochastic calculus), Marcus & Rosen [154] (for Gaussian processes), Peres & Mörters [162] (for random fractals), Chung [31] or Port & Stone [182] (for potential theory) or Blumenthal & Getoor [18] (for Markov processes) to name but a few.

Things the readers are expected to know: Our presentation is basically self-contained, starting from 'scratch' with continuous-time stochastic processes. We do, however, assume some basic measure theory (as in [204]) and a first course on probability theory and discrete-time martingales (as in [108] or [232]). Some 'remedial' material is collected in the appendix, but this is really intended as a back-up.

How to read this book: Of course, nothing prevents you from reading it linearly. But there is more material here than one could cover in a one-semester course. Depending on your needs and likings, there are at least three possible selections: *BM and Itô calculus*, *BM and its sample paths* and *BM as a Markov process*. The diagram on page xiii will give you some ideas how things depend on each other and how to construct your own 'Brownian sample path' through this book.

Whenever special attention is needed and to point out traps & pitfalls, we have used the ! sign in the margin. Also in the margin, there are cross-references to exercises at the end of each chapter which we think fit (and are sometimes needed) at that point.[1] They are not just drill problems but contain variants, excursions from and extensions of the material presented in the text. The proofs of the core material do not seriously depend on any of the problems.

! Ex. n.m

Writing an introductory text also meant that we had to omit many beautiful topics. Often we had to stop at a point where we, hopefully, got you really interested... Therefore, we close every chapter with a brief outlook on possible texts for further reading.

Many people contributed towards the completion of this project: First of all the students who attended our courses and helped – often unwittingly – to shape the presentation of the material. We profited a lot from comments by Niels Jacob (Swansea) and Panki Kim (Seoul National University) who used an early draft of the manuscript in one of his courses. Special thanks go to our colleagues and students Björn Böttcher, Katharina Fischer, Julian Hollender, Felix Lindner and Michael Schwarzenberger who read substantial parts of the text, often several times and at various stages. They found countless misprints, inconsistencies and errors which we would never have spotted. Björn helped out with many illustrations and, more importantly, contributed Chapter 22 on simulation. Finally we thank our colleagues and friends at TU Dresden and our families who contributed to this work in many uncredited ways. We hope that they approve of the result.

Dresden, February 2012

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1 For the readers' convenience there is a web page where additional material and solutions are available. The URL is http://www.motapa.de/brownian_motion/.