Contents

I	**Measures**		1
	1	Introduction	1
	2	Sigma-algebras	4
	3	Measures	9
	4	Uniqueness of measures	14
	5	Extension and construction of measures	19
II	**Integration**		29
	6	Measurable maps	29
	7	Borel functions	35
	8	The integral of positive measurable functions	43
	9	The integral of measurable functions	49
	10	Null sets	52
III	**Important theorems for integrals**		56
	11	Convergence theorems	56
	12	Parameter-dependent integrals	59
	13	Riemann vs. Lebesgue	64
	14	The Lebesgue spaces $\mathcal{L}^p$ and L^p	68
IV	**Products of measure spaces. Radon–Nikodým**		77
	15	Product measures	77
	16	The theorems of Fubini and Tonelli	83
	17	Integrals for image measures and convolutions	90
	18	The Radon-Nikodým theorem	100
	19	Products of infinitely many measure spaces	105
V	**Elementary probability**		111
	20	Hasard, chance and probability	111
	21	(Very) basic combinatorics	116
	22	Discrete probability distributions	123
	23	Continuous probability distributions	131
	24	Conditional probability	136

VI Independence 146
25 Independent events and random variables 146
26 Construction of (independent) random variables 155
27 Characteristic functions 163
28 Three classic limit theorems 172
29 Convergence of random variables 180
30 Characteristic functions and convergence in distribution 192
31 Convergence of independent random variables 196
32 The strong law of large numbers 203
33 Sums of independent random variables 208

VII Conditioning 216
34 Conditional expectation 216
35 Conditioning on $\mathscr{F} = \sigma(Y)$ 226

VIII Gaussian distributions and the Lindeberg-Lévy CLT 234
36 The multivariate normal law 234
37 The central limit theorem (CLT) 238

IX Martingales 249
38 Discrete martingales 249
39 Stopping 257
40 The martingale convergence theorem 262
41 L^2-martingales 265
42 Uniform integrability 269
43 Uniformly integrable martingales 274
44 Basic inequalities 278
45 Martingale proofs of some classical results 283
46 Martingales in continuous time 290

X Poisson Processes 300
47 Two special probability distributions 300
48 The Poisson process 306
49 PPs, Markov processes and martingales 315
50 Superpositition, thinning and colouring of PPs 322

XI Markov Chains 328
51 Random walks on the lattice $\mathbb{Z}^d$ 328
52 Finite Markov chains 338
53 The scope of Markov chains 350
54 The (strong) Markov property 356
55 Enter, hit, run and return 363
56 General random walks and recurrence 380
57 The Chung–Fuchs criterium 389

XII Brownian Motion 397
58 First steps towards Brownian motion 397
59 Existence of Brownian motion 399

60 BM as a martingale 406
61 How regular is a BM? 410
62 The Markov Property (MP) 415
63 Strong Markov Property (SMP) 419
64 The Reflection Principle 423
Name and subject index 432