Preface

The material is a honest rendering of my lecture notes of the courses »Measure and Integration« (3 contact hours §§1–19), »Introduction to probability« (4 contact hours, §§20–34, 38–40), »Further probability theory: Discrete random processes« (3–4 contact hours, §§35, 47–57) and »Probability with martingales« (3–4 contact hours, §§35–37, 41–46, 58–64) at TU Dresden. It is an introduction to measure and integration – suitable both for analysts and probabilists – and to probability theory & random processes up to the construction and first properties of Brownian motion.

The text is suitable for BSc students who have had a rigorous course in linear algebra and ϵ-δ-analysis. Some basic knowledge of functional analysis is helpful. The textbooks by Lang [16] (for linear algebra) and Rudin [27] (for analysis), [28, Chapters 4, 5] (for functional analysis) should be more than sufficient. For additional reference, I recommend Alt [2] (functional analysis) and the wonderful first chapter *Operator theory in finite-dimensional vector spaces* in Kato's book [14].

These notes contain the bare necessities. A more thorough treatment and plenty of exercises can be found in my books *Measures, Integrals and Martingales* [MIMS] and *Counterexamples in Measure and Integration* (with F. Kühn) [CEX] and my German-language textbooks *Maß und Integral* [MI], *Wahrscheinlichkeit* [WT] and *Martingale und Prozesse* [MaPs]. More on Brownian motion and stochastic (Itô) calculus is in *Brownian Motion. An introduction to stochastic processes* [BM] (with L. Partzsch). I tried to be as close as possible to the original lectures. Some essential material, which is usually set as (guided) exercise in the problem classes, is added as »starred items« like **Theorem**★. Handouts are either set in small print (if they appear in the running text) or contained in chapter appendices.

The subtitle »the theoretical minimum« alludes to the physicist Lev Landau, who *developed a comprehensive exam called the "Theoretical Minimum" which students were expected to pass before admission to the school* `https://en.wikipedia.org/wiki/Kharkiv_Theoretical_Physics_School` (accessed 30/Oct/2020).

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Abbreviations and symbols

Here I list the most common abbreviations and symbolic notation used throughout the text.

✎	this indicates that you should check it by yourself
⚠	this indicates a warning
💬	this indicates further information
a.a.	almost all
a.e.	almost every(where)
a.s.	almost surely
bdd	bounded
BL, cBL	(conditional) Beppo Levi theorem
BM	Brownian motion
b/o	because of
c.f.	characteristic function
cf.	confer, see
CLT	central limit theorem
d-convergence	convergence in distribution
DCT, cDCT	(conditional) domitated convergence theorem
e.g.	exempli gratia, for example
fdd	finite dimensional distributions
iid	independent identically distributed
LLN	law of large numbers
mble	measurable
MC	Markov chain
MCT	monotone class theorem
	or martingale convergence theorem
mg	martingale
MP	Markov property
$\mathbb{P}$-convergence	convergence in probability
PP, cPP	Poisson process, compound poisson process
rv, rvs	random variable, random variables

RW	random walk
SLLN	strong law of large numbers
SMP	strong Markov property
SRW	simple random walk
ui	uniformly integrable
WLLN	weak law of large numbers
wlog	without loss of generality
positive	always used in the sense »$\geqslant 0$«
negative	always used in the sense »$\leqslant 0$«
$\mathbb{N}$	natural numbers $1, 2, 3, \dots$
$\mathbb{N}_0$	natural numbers $0, 1, 2, 3, \dots$
$X \sim \mu$	the rv X is distributed like μ
$X \sim Y$	the rv X is distributed like the rv Y
$X \perp\!\!\!\perp Y$	X and Y are independent
$x \gg 1$, $\epsilon \ll 1$	x is sufficiently large, $\epsilon \in (0, 1)$ is sufficiently small
$\mathcal{L}^p(\dots)$, $L^p(\dots)$	Lebesgue spaces of integrable functions $1 \leqslant p \leqslant \infty$
$\mathcal{L}^0(\mathscr{A})$	$\mathscr{A}$-measurable functions
$\mathcal{E}(\mathscr{A})$	$\mathscr{A}$-measurable simple (»step«) functions

A »+« as sub- or superscript, such as $\mathcal{E}^+$ or $\mathcal{L}^p_+$ means the positive ($\geqslant 0$) elements of $\mathcal{E}$ or $\mathcal{L}^p$ etc.