Preface

Behind every decent Markov process there is a family of Lévy processes. Indeed, let $(X_t)_{t\geqslant 0}$ be a Markov process with state space $\mathbb{R}^d$ and assume, for the moment, that the limit

$$\lim_{t\to 0}\frac{1-\mathbb{E}^x e^{i\xi\cdot(X_t-x)}}{t} = q(x,\xi) \quad \forall x,\xi\in\mathbb{R}^d \tag{$\bigstar$}$$

exists such that the function $\xi\mapsto q(x,\xi)$ is continuous. We will see below that this is enough to guarantee that $q(x,\cdot)$ is, for each $x\in\mathbb{R}^d$, the characteristic exponent of a Lévy process; as such, it enjoys a Lévy–Khintchine representation

$$q(x,\xi) = -il(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}}\left(1-e^{iy\cdot\xi}+i\xi\cdot y\mathbb{1}_{[-1,1]}(|y|)\right)N(x,dy)$$

where $(l(x),Q(x),N(x,dy))$ is for every fixed $x\in\mathbb{R}^d$ a Lévy triplet. The function $q:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ is called the **symbol of the process**. The processes which admit a symbol behave locally like a Lévy process, and their infinitesimal generators resemble the generators of Lévy processes *with variable, i.e. x-dependent, coefficients*—this justifies the name **Lévy-type processes**. This guides us to the main topics of the present tract:

Characterization:	For which Markov processes does the limit ($\bigstar$) exist?
Construction:	Is there a Lévy-type process with a given symbol $q(x,\xi)$? Is there a 1-to-1 correspondence between symbols and processes?
Sample paths:	Can we use the symbol $q(x,\xi)$ in order to describe the sample path behaviour of the process?
Approximation:	Is it possible to use $q(x,\xi)$ to approximate and to simulate the process?

Let us put this point of view into perspective by considering first of all some d-dimensional Lévy process $(X_t)_{t\geqslant 0}$. Being a (strong) Markov process, $(X_t)_{t\geqslant 0}$ can be described by the transition function $p_t(x,dy)=\mathbb{P}^x(X_t\in dy)=\mathbb{P}(X_t+x\in dy)$

which, in turn, is uniquely characterized by the characteristic function

$$\mathbb{E}^x e^{i\xi\cdot X_t} = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\, p_t(x,dy) = e^{i\xi\cdot x} e^{-t\psi(\xi)} \tag{1}$$

and the characteristic exponent ψ. Thus,

$$1 - \int_{\mathbb{R}^d} e^{i\xi\cdot(y-x)}\, p_t(x,dy) = t\psi(\xi) + \mathrm{o}(t) \quad \text{as } t \to 0 \tag{2}$$

and, with some effort, we can derive from this the Lévy–Khintchine representation of the exponent ψ

$$\psi(\xi) = -il\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{iy\cdot\xi} + i\xi\cdot y\mathbb{1}_{[-1,1]}(|y|)\right)\nu(dy) \tag{3}$$

where (l,Q,ν) is the Lévy triplet. The key observation is that the family of measures $t^{-1}p_t(x,B+x) = t^{-1}\,\mathbb{P}(X_t \in B)$ converges[1] to the Lévy measure $\nu(B)$ as $t \to 0$ for all Borel sets $B \subset \mathbb{R}^d \setminus \{0\}$ satisfying $\nu(\bar{B}\setminus B^\circ) = 0$ and $0 \notin \bar{B}$.

In our calculation there is only one place where we used Lévy processes: The second equality sign in (1) which is the consequence of the translation invariance (spatial homogeneity) and infinite divisibility of a Lévy process. If we do away with it, and if we only assume that $(X_t)_{t\geqslant 0}$ is strong Markov with transition function $(p_t(x,dy))_{t\geqslant 0, x\in\mathbb{R}^d}$ we still have that

$$\lambda_t(x,\xi) := \mathbb{E}^x e^{i\xi\cdot(X_t-x)} = \int_{\mathbb{R}^d} e^{i\xi\cdot(y-x)}\, p_t(x,dy). \tag{1$'$}$$

Assume we *knew* that $t^{-1}p_t(x,B+x)$ has, as $t\to 0$, for every $x \in \mathbb{R}^d$ and suitable Borel sets $B \subset \mathbb{R}^d\setminus\{0\}$, a limit $N(x,B)$ which is a kernel on $\mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d\setminus\{0\})$. Then we would get, as in (2),

$$1 - \lambda_t(x,\xi) = -tq(x,\xi) + \mathrm{o}(t) \quad \text{as } t \to 0. \tag{2$'$}$$

But, what can be said about $q(x,\xi)$?

With some elementary harmonic analysis this can be worked out. Since $\xi \mapsto \lambda_t(x,\xi)$ is a characteristic function, it is continuous and positive definite (see page 50 for the definition); and since $1 \geqslant \lambda_t(x,0) \geqslant |\lambda_t(x,\xi)|$, we get from this that

$$\sum_{j,k=1}^{n} \left(\lambda_t(x,\xi_j - \xi_k) - 1\right)\mu_j\bar{\mu}_k \geqslant 0 \tag{4}$$

for all $n \geqslant 0$, $\xi_1,\ldots,\xi_n \in \mathbb{R}^d$ and $\mu_1,\ldots,\mu_n \in \mathbb{C}$ with $\sum_{j=1}^n \mu_j = 0$. This means that $\xi \mapsto \lambda_t(x,\xi) - 1$ is continuous and *conditionally positive definite*[2] (because of the *condition* $\sum_j \mu_j = 0$, cf. page 51). The important point is now that every

[1] The classical proofs use here Lévy's continuity theorem or the Helly–Bray theorem.

[2] also known as *negative definite*, and we will prefer this notion in the sequel, cf. Section 2.2

continuous and conditionally positive definite function enjoys a Lévy–Khintchine representation.

Obviously, inequality (4) remains valid if we divide by t and let $t \to 0$, so

$$\lim_{t\to 0} \frac{1 - \mathbb{E}^x e^{i\xi\cdot(X_t - x)}}{t} = q(x,\xi) \quad \forall x,\xi \in \mathbb{R}^d \tag{$\bigstar$}$$

defines a conditionally positive definite function $\xi \mapsto q(x,\xi)$. If it is also continuous, then it has for every fixed $x \in \mathbb{R}^d$ a Lévy–Khintchine representation, and each $q(x,\cdot)$ is the characteristic exponent of a Lévy process. We will call the function $q(x,\xi)$ the **symbol** of the process $(X_t)_{t\geqslant 0}$. In this sense it is correct to say that *behind every decent Markov process* $(X_t)_{t\geqslant 0}$ *there is a family of Lévy processes* $(L_t^{(x)})_{t\geqslant 0, x\in\mathbb{R}^d}$ whose characteristic exponents are given by ($\bigstar$), and we are back at the point where we started our discussion.

Sufficient conditions for the limit ($\bigstar$) to exist are best described by a list of Lévy-type processes: Lévy processes, of course, (cf. Section 2.1), any Feller process whose infinitesimal generator has a sufficiently rich domain (Sections 2.3, 2.4), many Lévy-driven stochastic differential equations (Section 3.2), or temporally homogeneous Markovian jump-diffusion semimartingales (Section 2.5) provided that their extended generator contains sufficiently many functions. As it turns out, the symbol $q(x,\xi)$ encodes, via its Lévy–Khintchine representation and the (necessarily) x-dependent Lévy triplet, the semimartingale characteristics of the stochastic process; moreover, it yields a simple representation of the infinitesimal generator as a pseudo-differential operator.

We are not aware of necessary conditions such that ($\bigstar$) defines a negative definite symbol, although the class of temporally homogeneous Markovian jump-diffusion semimartingales looks pretty much to be the largest class of decent strong Markov processes admitting a symbol.

Most of our results hold for any *decent* strong Markov process admitting a symbol $q(x,\xi)$, but we restrict our attention to Feller processes where *decency* comes from the natural assumption that the compactly supported smooth functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the infinitesimal generator. The key results in this direction are our short proof of the Courrège–von Waldenfels theorem (Theorem 2.21) and the probabilistic formula for the symbol, Theorems 2.36 and 2.44.

Let us briefly explain how the material is organized. The *Primer on Feller Semigroups*, Chapter 1, is included in order to make the material accessible for the novice, but also to serve as a reference. For the more experienced reader, the ideal point of departure should be Section 2.1 on Lévy processes which leads directly to the characterization of Feller processes. Among the central results of Chapter 2 is the characterization of the generators as pseudo-differential operators and the fact that Feller processes are semimartingales: In both cases the symbol $q(x,\xi)$ and its x-dependent Lévy triplet are instrumental. Chapter 3 is devoted to various construction methods for Feller processes. This is probably the most technical part of our treatise since techniques from different areas of mathematics come to bearing; it is already difficult to describe the results, to present complete proofs in this essay is near impossible. Nevertheless we tried to describe the ideas how things fit together, and we hope that the interested reader follows up on the references provided. Perturbations and time-changes for Feller processes are briefly discussed in Chapter 4. In particular, we obtain conditions such that the Feller property is preserved under these transformations. From Chapter 5 onwards, things become more probabilistic: Now we show how to use the symbol $q(x,\xi)$ in order to describe the behaviour of the sample paths of a Feller process. For Lévy processes this approach has a long tradition starting with the papers by Blumenthal–Getoor [32, 33] in the early sixties; a survey is given in Fristedt [113]. The principal tool for these investigations are probability estimates for the running maximum of the process in terms of the symbol (Section 5.1). Using these estimates we can define Blumenthal–Getoor–Pruitt indices for Feller processes which, in turn, allow us to find bounds for the Hausdorff dimension of the sample paths, describe the (polynomial) short- and long-time asymptotics of the paths, their p-variation, their Besov regularity etc. Returning to the level of transition semigroups we then investigate global properties (in the sense of Fukushima et al.) in Chapter 6. We focus on functional inequalities and their stability under subordination and on coupling methods; the latter are explained in detail for Lévy- and linear Ornstein–Uhlenbeck processes. The classical topics of transience and recurrence are discussed from the perspective of Meyn and Tweedie, with an emphasis on stable-like processes. In the final Chapter 7 we show how the viewpoint of a Feller process being locally Lévy can be used to approximate the sample paths of Feller processes. This allows us, for the first time, to simulate Feller processes with unbounded coefficients. We close this treatise with a list of open problems which we think are important for the further development of the subject.

We cannot cover all aspects of Lévy-type processes in this survey. Notable omissions are probabilistic potential theory, the general theory of Dirichlet forms, heat kernel estimates and processes on domains. Our choice of material was, of course, influenced by personal liking, by our own research interests and by the desire to have a clear focus. Some topics, e.g. probabilistic potential theory and Dirichlet forms, are more naturally set in the wider framework of general Markov processes and there are, indeed, monographs which we think are hard to match: In potential theory there are Chung's books [71, 72] (for Feller processes), Sharpe [299] (for general Markov processes), Port–Stone [239] and Bertoin [28] (for Brownian motion and Lévy processes) and for Dirichlet forms there is Fukushima et al. [116] (for the symmetric case), and Ma–Röckner [212] (for the non-symmetric case). Heat kernel estimates are usually discussed in an L^2-framework, cf. Chen [67] for an excellent survey, or for various perturbations of stable Lévy processes (also on domains), e.g. as in Chen–Kim–Song [68, 69]. An interesting geometric approach has recently been proposed by [162]. Finally, processes on domains with general Wentzell boundary conditions [352] for the generator are still a problem: While some progress has been made in the one-dimensional case (cf. Mandl [217], Langer and co-workers [200, 201]), the multidimensional case is wide open, and the best treatment is Taira [314].

A few words on the style of this treatise are in order. Some time ago, we have been invited to contribute a survey paper to the *Lévy Matters* subseries of the Springer Lecture Notes in Mathematics, updating the earlier paper *Lévy-type processes and pseudo-differential operators* by N. Jacob and one of the present authors. Soon, however, it became clear that the developments in the past decade have been quite substantial while, on the other hand, much of the material is scattered throughout the literature and that a comprehensive treatise on Feller processes is missing. With this essay we try to fill this gap, providing a reliable source for reference (especially for those elusive *folklore* results), making a technically demanding area easily accessible to future generations of researchers and, at the same time, giving a snapshot of the state-of-the-art of the subject. Just as one would expect in a survey, we do not always (want to) give detailed proofs, but we provide precise references whenever we omit proofs or give only a rough outline of the argument (sometimes also sailing under the nickname 'proof'). On the other hand, quite a few theorems are new or contain substantial improvements of known results, and in all those cases we do include full proofs or describe the necessary changes to the literature. We hope that the exposition is useful for and accessible to anyone with a working knowledge of Lévy- or continuous-time Markov processes and some basic functional analysis.

It is a pleasure to acknowledge the support of quite a few people. Niels Jacob has our best thanks, his ideas run through the whole text, and we shall think it a success if it pleases him.

Without the named (and, as we fear, often unnamed) contributions of our coauthors and fellow scientists such a survey would not have been possible; we are grateful that we can present and build on their results. Anite Behme, Xiaoping Chen, Katharina Fischer, Julian Hollender, Victorya Knopova, Franziska Kühn, Huaiqian Li, Felix Lindner, Michael Schwarzenberger, and Nenghui Zhu read substantial portions of various β-versions of this survey, pointed out many mistakes and inconsistencies, and helped us to improve the text; the examples involving affine processes were drafted by Michael Schwarzenberger.

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Finally, we thank our friends and families who—we are pretty sure of it—are more than happy that this work has come to an end.

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