

Index

This should be used in conjunction with the bibliography and the list of symbols. Numbers following entries are page numbers which, if accompanied by (Pr $n.m$), refer to Problem $n.m$ on that page; a number with a trailing ‘n’ indicates that a footnote is being referenced. Unless stated otherwise ‘integral’, ‘integrability’, etc. always refer to the (abstract) Lebesgue integral. Within the index we use ‘L-...’ and ‘R-...’ as a shorthand for ‘(abstract) Lebesgue-...’ and ‘Riemann-...’

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- Young inequality, 133 (Pr 13.6)

List of Symbols

This is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below. Numbers following entries are page numbers, with the occasional (Pr $m.n$) referring to Problem $m.n$ on the respective page.

Unless stated otherwise, binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leq g$, $f < g$ or limiting relations $f_n \xrightarrow{n \rightarrow \infty} f$, $\lim_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$, $\sup_i f_i$ or $\inf_i f_i$ are always understood pointwise.

Alternatives are indicated by square brackets, i.e., ‘if $A [B] \dots$ then $P [Q]$ ’ should be read as ‘if $A \dots$ then P ’ and ‘if $B \dots$ then Q ’.

Generalities

positive	always in the sense ≥ 0
negative	always in the sense ≤ 0
\mathbb{N}	natural numbers: 1, 2, 3, ...
\mathbb{N}_0	positive integers: 0, 1, 2, ...
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integer, rational, real, complex numbers
$\overline{\mathbb{R}}$	$[-\infty, +\infty]$
$\inf \emptyset, \sup \emptyset$	$\inf \emptyset = +\infty, \sup \emptyset = -\infty$
$a \vee b$	maximum of a and b
$a \wedge b$	minimum of a and b
$\liminf_n a_n$	$\sup_k \inf_{n \geq k} a_n$, 409
$\limsup_n a_n$	$\inf_k \sup_{n \geq k} a_n$, 409
$ x $	Euclidean norm in \mathbb{R}^n , $ x ^2 = x_1^2 + \dots + x_n^2$
$\langle x, y \rangle$	scalar product $\sum_{i=1}^n x_i y_i$
ω_n	volume of the unit ball in \mathbb{R}^n , 181

Sets and set operations

$A \cup B$	union 6
$A \cup B$	union of disjoint sets, 6
$A \cap B$	intersection, 6
$A \setminus B$	set-theoretic difference, 6
A^c	complement of A , 6
$A \Delta B$	$(A \setminus B) \cup (B \setminus A)$
$A \subset B$	subset (includes ‘=’), 6
$A \subsetneq B$	proper subset, 6
$A \times B$	Cartesian product
\overline{A}	closure of A
A°	open interior of A
$A_n \uparrow A, A_n \downarrow A$	23

$A \times B$	Cartesian product
A^n	n -fold Cartesian product
$A^{\mathbb{N}}$	infinite sequences in A
$\#A$	cardinality of A , 8
$t \cdot A$	$\{ta : a \in A\}$
$x + A$	$\{x + a : a \in A\}$
$E \cap \mathcal{A}$	$\{E \cap A : A \in \mathcal{A}\}$, 17
$\liminf_n A_n$	$\bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} A_n$, 413
$\limsup_n A_n$	$\bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} A_n$, 413
$(a, b), [a, b]$	open, closed intervals
$(a, b], [a, b)$	half-open intervals
$B_r(x)$	open ball with radius r and centre x

Families of sets

\mathcal{A}	generic σ -algebra, 16
$\overline{\mathcal{A}}$	completion 30 (Pr 4.15)
$(\mathcal{A}_i)_{i \in I}$	filtration, 276
\mathcal{A}_∞	$\sigma(\mathcal{A}_i : i \in I) = \sigma(\bigcup_{i \in I} \mathcal{A}_i)$
$\mathcal{A}_{-\infty}$	$\bigcap_{\ell \in -\mathbb{N}} \mathcal{A}_\ell$
$\mathcal{A}_\tau, \mathcal{A}_\sigma$	283
$\mathcal{A} \times \mathcal{B}$	$\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ rectangles
$\mathcal{A} \otimes \mathcal{B}$	product- σ -algebra, 138
$\mathcal{B}(X)$	Borel sets in X , 18: $X = \mathbb{R}^n$ (18), $X = A \subset \mathbb{R}^n$ (22 Pr 3.13), $X = \overline{\mathbb{R}}$ (61), $X = \mathbb{C}$ (88 Pr 10.9, 423)
$\overline{\mathcal{B}(\mathbb{R}^n)}$	completion of the Borel sets, 151 (Pr 14.15), 172, 429

$\mathcal{I}, \mathcal{I}^o, \mathcal{I}^{\text{rat}}$ rectangles in \mathbb{R}^n , 19
 \mathcal{N}_μ μ -null sets, 29 (Pr 4.12), 89
 $\mathcal{O}(X)$ topology, open sets in X , 18
 $\mathcal{P}(X)$ all subsets of X , 13
 $\delta(\mathcal{G})$ Dynkin system generated by \mathcal{G} , 32
 $\sigma(\mathcal{G})$ σ -algebra generated by \mathcal{G} , 17
 $\sigma(T), \sigma(T_i: i \in I)$ σ -algebra generated by the map(s) T , resp., T_i , 55
 (X, \mathcal{A}) measurable space, 23
 (X, \mathcal{A}, μ) measure space, 24
 $(X, \mathcal{A}, \mathcal{A}_i, \mu)$ filtered measure space, 275, 276

Measures and integrals


μ, ν generic measures
 δ_x Dirac measure in x , 26
 λ, λ^n Lebesgue measure, 27
 $\mu \circ T^{-1}, T(\mu)$ image measure, 55
 $u \cdot \mu, u\mu$ measure with density, 86
 $\mu \times \nu$ product of measures, 141
 $\mu * \nu$ convolution, 157
 $\mu \ll \nu$ absolute continuity, 300
 $\mu \perp \nu$ singular measures, 306
 $d\nu/d\mu$ Radon–Nikodým derivative, 230, 301
 $\mathbb{E}^{\mathcal{G}}$ conditional expectation, 343, 346, 348
 $\int u d\mu$ 74, 82
 $\int_A u d\mu$ $\int \mathbb{1}_A u d\mu$, 86
 $\int u dx$ 83
 $\int u dT(\mu)$ $\int u \circ T d\mu$, 154
 $\int_a^b u(x) dx, (R) \int_a^b u(x) dx$ Riemann integral, 102, 443

Functions and spaces

$\mathbb{1}_A$ $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$
 $\text{sgn}(x)$ $\mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$
 id identity map or matrix
 \det determinant (of a matrix)
 $u(A)$ $\{u(x) : x \in A\}$
 $u^{-1}(\mathcal{B})$ $\{u^{-1}(B) : B \in \mathcal{B}\}$, 17
 u^+ $\max\{u(x), 0\}$ positive part
 u^- $-\min\{u(x), 0\}$ negative part

$\{u \in B\}$ $\{x : u(x) \in B\}$, 60
 $\{u \geq \lambda\}$ $\{x : u(x) \geq \lambda\}$ etc., 60
 $\text{supp } u$ support $\{\overline{u \neq 0}\}$
 $C(X)$ continuous functions on X
 $C_b(X)$ bounded continuous functions on X
 $C_\infty(X)$ continuous functions on X with $\lim_{|x| \rightarrow \infty} u(x) = 0$
 $C_c(X)$ continuous functions on X with compact support
 $\mathcal{E}(\mathcal{A})$ simple functions, 63
 $\mathcal{M}(\mathcal{A})$ measurable functions, 62
 $\mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ measurable functions, values in $\overline{\mathbb{R}}$, 62
 \mathcal{L}^1 integrable functions, 82
 $\mathcal{L}^1_{\overline{\mathbb{R}}}$ integrable functions, values in $\overline{\mathbb{R}}$, 82
 $\mathcal{L}^p, \mathcal{L}^\infty$ 116
 L^p, L^∞ 119
 $\mathcal{L}^p_{\mathbb{C}}, L^p_{\mathbb{C}}$ 423, 214
 $\ell^1(\mathbb{N}), \ell^p(\mathbb{N})$ 124–125
 $\|u\|_p$ $(\int |u|^p d\mu)^{1/p}, p < \infty$, 116
 $\|u\|_\infty$ $\inf\{c : \mu\{|u| \geq c\} = 0\}$, 114
 $\mathfrak{M}_r^+(X)$ regular measures on X , 437

Abbreviations

a.a., a.e. almost all/every(where), 89
 UI uniformly integrable, 258
 w.r.t. with respect to
 \cup/\cap -stable stable under finite unions/intersections
 \square end of proof
 \square indicates that a small intermediate step is required
 * can be omitted without loss of continuity
 (in the margin) caution
 (D_1) – (D_3) Dynkin system, 32, 37
 (M_0) – (M_3) measure, 23
 (OM_1) – (OM_3) outer measure, 40, 200
 (O_1) – (O_3) topology, 18
 (S_1) – (S_3) semi-ring, 39
 (Σ_1) – (Σ_3) σ -algebra, 16

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