Index

This should be used in conjunction with the bibliography and the list of symbols. Numbers following entries are page numbers which, if accompanied by (Pr *n.m*), refer to Problem *n.m* on that page; a number with a trailing 'n' indicates that a footnote is being referenced. Unless stated otherwise 'integral', 'integrability', etc. always refer to the (abstract) Lebesgue integral. Within the index we use 'L-...' and 'R-...' as a shorthand for '(abstract) Lebesgue-...' and 'Riemann-...'

List of Symbols

This is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below. Numbers following entries are page numbers, with the occasional (Pr *m.n*) referring to Problem *m.n* on the respective page.

Unless stated otherwise, binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limiting relations $f_n \xrightarrow[n\to\infty]{} f$, $\lim_n f_n$, $\liminf_n f_n$, $\limsup f_n$, $\sup_i f_i$ or $\inf_i f_i$ are always understood pointwise.

Alternatives are indicated by square brackets, i.e., '*if A* [*B*] ... *then P* [*Q*]' should be read as '*if A* ... *then P*' and '*if B* ... *then Q*'.

Generalities

positive	always in the sense $\geqslant 0$
negative	always in the sense $\leqslant 0$
$\mathbb{N}$	natural numbers: $1, 2, 3, \ldots$
$\mathbb{N}_0$	positive integers: $0, 1, 2, \ldots$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integer, rational, real, complex numbers
$\overline{\mathbb{R}}$	$[-\infty, +\infty]$
$\inf \emptyset, \sup \emptyset$	$\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$
$a \vee b$	maximum of a and b
$a \wedge b$	minimum of a and b
$\liminf_n a_n$	$\sup_k \inf_{n \geqslant k} a_n$, 409
$\limsup_n a_n$	$\inf_k \sup_{n \geqslant k} a_n$, 409
$\lvert x \rvert$	Euclidean norm in $\mathbb{R}^n$, $\lvert x \rvert^2 = x_1^2 + \cdots + x_n^2$
$\langle x, y \rangle$	scalar product $\sum_{i=1}^n x_i y_i$
ω_n	volume of the unit ball in $\mathbb{R}^n$, 181

Sets and set operations

$A \cup B$	union 6
$A \dot\cup B$	union of disjoint sets, 6
$A \cap B$	intersection, 6
$A \setminus B$	set-theoretic difference, 6
A^c	complement of A, 6
$A \triangle B$	$(A \setminus B) \cup (B \setminus A)$
$A \subset B$	subset (includes '='), 6
$A \subsetneq B$	proper subset, 6
$A \times B$	Cartesian product
$\overline{A}$	closure of A
A°	open interior of A
$A_n \uparrow A$, $A_n \downarrow A$	23

$A \times B$	Cartesian product
A^n	n-fold Cartesian product
$A^{\mathbb{N}}$	infinite sequences in A
$\#A$	cardinality of A, 8
$t \cdot A$	$\{ta : a \in A\}$
$x + A$	$\{x + a : a \in A\}$
$E \cap \mathscr{A}$	$\{E \cap A : A \in \mathscr{A}\}$, 17
$\liminf_n A_n$	$\bigcup_{k\in\mathbb{N}} \bigcap_{n\geqslant k} A_n$, 413
$\limsup_n A_n$	$\bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant k} A_n$, 413
$(a, b), [a, b]$	open, closed intervals
$(a, b], [a, b)$	half-open intervals
$B_r(x)$	open ball with radius r and centre x

Families of sets

$\mathscr{A}$	generic σ-algebra, 16
$\overline{\mathscr{A}}$	completion 30 (Pr 4.15)
$(\mathscr{A}_i)_{i\in I}$	filtration, 276
$\mathscr{A}_\infty$	$\sigma(\mathscr{A}_i : i \in I) = \sigma\left(\bigcup_{i\in I} \mathscr{A}_i\right)$
$\mathscr{A}_{-\infty}$	$\bigcap_{\ell\in-\mathbb{N}} \mathscr{A}_\ell$
$\mathscr{A}_\tau$, $\mathscr{A}_\sigma$	283
$\mathscr{A} \times \mathscr{B}$	$\{A \times B : A \in \mathscr{A},\ B \in \mathscr{B}\}$ rectangles
$\mathscr{A} \otimes \mathscr{B}$	product-σ-algebra, 138
$\mathscr{B}(X)$	Borel sets in X, 18: $X = \mathbb{R}^n$ (18), $X = A \subset \mathbb{R}^n$ (22 Pr 3.13), $X = \overline{\mathbb{R}}$ (61), $X = \mathbb{C}$ (88 Pr 10.9, 423)
$\overline{\mathscr{B}(\mathbb{R}^n)}$	completion of the Borel sets, 151 (Pr 14.15), 172, 429

$\mathcal{J}, \mathcal{J}^o, \mathcal{J}_{\text{rat}}$ rectangles in $\mathbb{R}^n$, 19
$\mathcal{N}_\mu$ μ-null sets, 29 (Pr 4.12), 89
$\mathcal{O}(X)$ topology, open sets in X, 18
$\mathcal{P}(X)$ all subsets of X, 13
$\delta(\mathcal{G})$ Dynkin system generated by $\mathcal{G}$, 32
$\sigma(\mathcal{G})$ σ-algebra generated by $\mathcal{G}$, 17
$\sigma(T), \sigma(T_i : i \in I)$ σ-algebra generated by the map(s) T, resp., T_i, 55
$(X, \mathcal{A})$ measurable space, 23
$(X, \mathcal{A}, \mu)$ measure space, 24
$(X, \mathcal{A}, \mathcal{A}_i, \mu)$ filtered measure space, 275, 276

Measures and integrals

μ, ν generic measures
δ_x Dirac measure in x, 26
λ, λ^n Lebesgue measure, 27
$\mu \circ T^{-1}, T(\mu)$ image measure, 55
$u \cdot \mu, u\mu$ measure with density, 86
$\mu \times \nu$ product of measures, 141
$\mu \star \nu$ convolution, 157
$\mu \ll \nu$ absolute continuity, 300
$\mu \perp \nu$ singular measures, 306
$d\nu/d\mu$ Radon–Nikodým derivative, 230, 301
$\mathbb{E}^{\mathcal{G}}$ conditional expectation, 343, 346, 348
$\int u\, d\mu$ 74, 82
$\int_A u\, d\mu$ $\int \mathbb{1}_A u\, d\mu$, 86
$\int u\, dx$ 83
$\int u\, dT(\mu)$ $\int u \circ T\, d\mu$, 154
$\int_a^b u(x)\, dx$, $(R)\int_a^b u(x)\, dx$ Riemann integral, 102, 443

Functions and spaces

$\mathbb{1}_A$ $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$
$\operatorname{sgn}(x)$ $\mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$
id identity map or matrix
det determinant (of a matrix)
$u(A)$ $\{u(x) : x \in A\}$
$u^{-1}(\mathcal{B})$ $\{u^{-1}(B) : B \in \mathcal{B}\}$, 17
u^+ $\max\{u(x), 0\}$ positive part
u^- $-\min\{u(x), 0\}$ negative part
$\{u \in B\}$ $\{x : u(x) \in B\}$, 60
$\{u \geqslant \lambda\}$ $\{x : u(x) \geqslant \lambda\}$ etc., 60
$\operatorname{supp} u$ support $\overline{\{u \neq 0\}}$
$C(X)$ continuous functions on X
$C_b(X)$ bounded continuous functions on X
$C_\infty(X)$ continuous functions on X with $\lim_{|x|\to\infty} u(x) = 0$
$C_c(X)$ continuous functions on X with compact support
$\mathcal{E}(\mathcal{A})$ simple functions, 63
$\mathcal{M}(\mathcal{A})$ measurable functions, 62
$\mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ measurable functions, values in $\overline{\mathbb{R}}$, 62
$\mathcal{L}^1$ integrable functions, 82
$\mathcal{L}^1_{\overline{\mathbb{R}}}$ integrable functions, values in $\overline{\mathbb{R}}$, 82
$\mathcal{L}^p, \mathcal{L}^\infty$ 116
L^p, L^∞ 119
$\mathcal{L}^p_{\mathbb{C}}, L^p_{\mathbb{C}}$ 423, 214
$\ell^1(\mathbb{N}), \ell^p(\mathbb{N})$ 124–125
$\|u\|_p$ $\left(\int |u|^p\, d\mu\right)^{1/p}$, $p < \infty$, 116
$\|u\|_\infty$ $\inf\{c : \mu\{|u| \geqslant c\} = 0\}$, 114
$\mathfrak{M}_\tau^+(X)$ regular measures on X, 437

Abbreviations

a.a., a.e. almost all/every(where), 89
UI uniformly integrable, 258
w.r.t. with respect to
$\cup/\cap$-stable stable under finite unions/intersections
□ end of proof
[✍] indicates that a small intermediate step is required
* can be omitted without loss of continuity
 (in the margin) caution

(D_1)–(D_3) Dynkin system, 32, 37
(M_0)–(M_3) measure, 23
(OM_1)–(OM_3) outer measure, 40, 200
$(\mathcal{O}_1)$–$(\mathcal{O}_3)$ topology, 18
(S_1)–(S_3) semi-ring, 39
(Σ_1)–(Σ_3) σ-algebra, 16

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