Measures, Integrals and Martingales

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by

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List of misprints and smaller changes to the present text (1st printing 2005, 2nd printing 2007). Last update: May 30, 2016.

PAGE, LINE	READS	SHOULD READ
p 20, Problem 3.5	be subsets of X	be nonempty subsets of X
p 20, Problem 3.5 (i)	no proper subset $B \subsetneq A$	no proper subset $\emptyset \neq B \subsetneq A$
p 20, Problem 3.9	Is this still true for the family $\mathbb{B}' := \{B_r(x) \,:\, x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$?	Denote by $B_r(x)$ an open ball in $\mathbb{R}^n$ with centre x and radius r. Show that the Borel sets $\mathcal{B}(\mathbb{R}^n)$ are generated by the family of open balls $\mathbb{B} := \{B_r(x) \,:\, x \in \mathbb{R}^n,\ r > 0\}$. Is this still true for the family $\mathbb{B}' := \{B_r(x) \,:\, x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$?
p 26, line 6 below	from Example 2.3(v)	from Example 3.3(v)
p 31, line 10 below	$D, E \in \mathcal{E}, D \cap E = \emptyset$	$D, E \in \mathcal{D}, D \cap E = \emptyset$
p 35, line 10 above	invoke T5.5	invoke T5.7
p 36, Problem 5.3	for all $A, B \in \mathcal{D}$	for all $A, B \in \mathcal{D}$ with $A \subset B$
p 36, Problem 5.8	Mimic the proof of Theorem 5.8(I)...	Mimic the proof of Theorem 5.8(i)...
p 45, lines 10–11 below	side-lengths $\lim_{j\to\infty} \big(b_k^{(j)} - b_k^{(j)}\big) > 0$	side-lengths $\lim_{j\to\infty} \big(b_k^{(j)} - a_k^{(j)}\big) > 0$
p 46, Problem 6.2 (i)	... that $\mu(N) = 0$ for all $N \subset Q \setminus A$ with $N \in \mathcal{A}$	... that $\mu(N) = 0$ for all $N \subset A \setminus Q$ with $N \in \mathcal{A}$
p 46, Problem 6.2 (i), Hint	... and $\mu(B) - \mu^*(Q) \leqslant 1/k$.	... and $\mu(B_k) - \mu^*(Q) \leqslant 1/k$.
p 54, Problem 7.5, line 13 below	is ‘take out‘ measurable if, and only if,...	is measurable if, and only if,...
p 54, Problem 7.9, line 2 below	Let μ be a measure on $(\mathbb{R}, \mathcal{B}^1)$.	Let μ be a measure on $(\mathbb{R}, \mathcal{B}^1)$ with $\mu[-n, n) < \infty$, $n \in \mathbb{N}$.
p 58, Lemma 8.2	$\mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$	$\mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}}) \stackrel{\text{def}}{=} \{A \cap \mathbb{R} \,:\, A \in \mathcal{B}(\bar{\mathbb{R}})\}$
p 62, line 14 above	$\emptyset$, if $\lambda < 0$	X, if $\lambda < 0$
p 65, Problem 8.8, hint	$\{f > \alpha\}$	$\{u > \alpha\}$
p 71, line 5 below	elementary	simple
p 76, line 10 above	elementary	simple

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PAGE, LINE	READS	SHOULD READ
p 78, Theorem 10.4, proof (i)	*line missing, add* $\longrightarrow$	.... by 9.8(ii). To see the integral formula we assume that $\alpha \leqslant 0$. Then $(\alpha u)^{\pm} = (-\alpha)(-u)^{\pm} = -\alpha u^{\mp}$ and the formula follows directly from Definition 10.1. The case $\alpha \geqslant 0$ follows in a similar way.
p 78, Theorem 10.4, proof (ii)	*line missing, add* $\longrightarrow$	.... by 9.8(iii). For the integral formula observe that $(u+v)^+ - (u+v)^- = u+v = (u^+ + v^+) - (u^- + v^-)$. Thus, $(u+v)^+ + u^- + v^- = (u+v)^- + u^+ + v^+$ and we can integrate this equality and use 9.8(ii) to get $$\int (u+v)^+\,d\mu + \int u^-\,d\mu + \int v^-\,d\mu = \int (u+v)^-\,d\mu + \int u^+\,d\mu + \int v^+\,d\mu.$$ Since all terms are finite, we can rearrange this equality and use Definition 10.1 to see $$\int (u+v)\,d\mu = \int u\,d\mu + \int v\,d\mu$$
p 81, line 1 above	$u \in \mathcal{L}^1_{\overline{\mathbb{R}}}(\mu)$ be a numerical integrable function	$u \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ be a numerical measurable function
p 82, line 9,10 above	$\overset{10.9(\mathrm{i})}{=}$ (twice)	$\overset{10.9(\mathrm{ii})}{=}$
p 83, line 5 above	3.4(iii')	4.4(iii')
p 85, Problem 10.9	$u \in \mathcal{L}^1(P) \iff \sum_{j=0}^{\infty} P(\{u \geqslant j\}) < \infty.$	$u \in \mathcal{L}^1(P) \iff \sum_{j=0}^{\infty} P(\{\lvert u\rvert \geqslant j\}) < \infty.$
p 86, Probl. 10.12(i)	$\mu_*(E) + \mu_*(F) \leqslant \mu_*(E \cup F)$	$\mu_*(E) + \mu_*(F) \leqslant \mu_*(E \uplus F)$
p 95, line 6 below	$x \in [a,b]$	$x \in [a,b] \setminus \Pi$
p 96, line 5 above	$\{\Sigma[f] = \sigma[f]\} \subset \{x : f(x) \text{ is continuous}\} \cup \Pi$,	$\{\Sigma[f] = \sigma[f]\} \supset \{x : f(x) \text{ is continuous}\}$,
p 102, Problem 11.15	Let X be a random variable on ...	Let X be a positive random variable on ...
p 111, lines 3–5 above	$\sum_{j=k+1}^{\infty}$	$\sum_{j=k}^{\infty}$ **(four instances!!)**
p 118, Problem 12.18	The conjugate index is given by $q := 1/(p-1) < 0$.	The conjugate index is given by $q := p/(p-1) < 0$.
p 122, line 11 below	$(A \times X) \cap (Y \times B)$	$(A \times Y) \cap (X \times B)$
p 127, line 11/12 below	$(A_1 \times Y) \cap (X \times A_2)$	$(A_1 \times X_2) \cap (X_1 \times A_2)$
p 128, line 1 above	$\mathcal{A}_j/\mathcal{A}_j$	$\mathcal{A}_2/\mathcal{A}_j$
p 133, Prob. 13.13(iv) line 14 below	$\Big[\phi(F(s)) - \phi(F(s-)) - \phi'(F(s))\Delta F(s)\Big]$	$\Big[\phi(F(s)) - \phi(F(s-)) - \phi'(F(s-))\Delta F(s)\Big]$
p 133, Problem 13.14, line 9 below	$\mu_f(\{f \geqslant t\})$	$\mu_f(t) := \mu(\{\lvert f\rvert \geqslant t\})$

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pp 134-5, Thm 14.1	**The proof of Theorem 14.1 shows more than what is claimed. Add, therefore, the following line inside the statement of Theorem 14.1 after formula (14.2):**	*In particular, $u\circ T$ is μ-integrable if, and only if, u is $T(\mu)$-integrable.*
p 135, lines 3–5 below	Often we are ... Theorem 14.1	**Delete lines 3–5 from below**
p 137, line 5 below	$\ldots = \alpha(\mu\times\nu)$.	$\ldots = \alpha(\mu\times\nu)(B)$.
p 138, line 7 above	$\mathcal{L}^1(\mu\times\nu)$	$\mathcal{L}^1(\mu\star\nu)$
p 139, line 13 above	such that $\|u-\phi_\epsilon\|$	such that $\|u-\phi_\epsilon\|_p$
p 141, Problem 14.7, line 11 above	$u\in\mathcal{L}^1(\lambda^1)$	$u\in\mathcal{L}^1_+(\lambda^1)$
p 141, Problem 14.9, line 17 above	$\mathcal{L}^p$, $\mathcal{L}^q$	$\mathcal{L}^p(\lambda^n)$, $\mathcal{L}^q(\lambda^n)$
p 141, Prob. 14.10(iii)	$\ldots = \{y\;:\;\forall x\in\operatorname{supp}u\;:\;\lvert x-y\rvert\leqslant\epsilon\}$	$\ldots = \{y\;:\;\exists x\in\operatorname{supp}u\;:\;\lvert x-y\rvert\leqslant\epsilon\}$
p 141, Prob. 14.11(ii)	$u\star w(x)=\ldots$	$v\star w(x)=\ldots$
p 141, Prob. 14.11, line 2 below	... commutativity of the convolution which was used in ...	... associativity of the convolution which is implicit in ...
p 155, line 3 below	$\lambda^2(dx_n)$	$\lambda^1(dx_n)$
p 160, Prob. 15.6(i), line 5 below	$\det D\Phi(x)$	$\lvert D\Phi(x)\rvert^2$
p 160, Prob. 15.6(ii), line 4 below	$\det D\Phi(x)$	$\lvert D\Phi(x)\rvert$
p 161, Prob. 15.6(iv), line 5 above	$\det D\tilde\Phi(x,r) = 1+(f'(x)^2) - rf''(x)$	$\det D\tilde\Phi(x,r) = \sqrt{1+(f'(x))^2} - \dfrac{r\,f''(x)}{1+(f'(x))^2}$
p 161, Prob. 15.6(vi), line 11 above	$\ldots\big\lvert\det D\tilde\Phi(x,r)\big\rvert\,\lambda^1(dr)$	$\ldots\big\lvert\det D\tilde\Phi(x,s)\big\rvert\,\lambda^1(ds)$
p 161, Prob. 15.6(vii), line 14 above	$\lim_{r\downarrow 0}\lambda^2(C(r)) = \int_{\Phi^{-1}(C)}\lvert\det D\Phi(x)\rvert\,\lambda^1(dx)$	$\lim_{r\downarrow 0}\dfrac{1}{2r}\lambda^2(C(r)) = \int_{\Phi^{-1}(C)}\lvert\det D\tilde\Phi(x,0)\rvert\,\lambda^1(dx)$
p 161, Prob. 15.7(ii), line 12 below	$\int_M u(r\,\xi)\,r^n\,d\lambda_M(\xi)=\ldots$	$\int_M u(r\,\xi)\,r^n\,d\lambda^n(\xi)=\ldots$
p 161, Prob. 15.7(iii), line 8 below	$=\int_{(0,\infty)}\int_{\{\lVert x\rVert=1\}} u(r\,x)\,\sigma(dx)\,\lambda^1(dr)$.	$=\int_{(0,\infty)}\int_{\{\lVert x\rVert=1\}} r^{n-1}\,u(r\,x)\,\sigma(dx)\,\lambda^1(dr)$.

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<td>pp 165-6, Proof of Theorem 16.6</td>
<td>(iii)⇒(ii): Since $\lim_{j\to\infty}\int|u_j|^p\,d\mu=\int|u|^p\,d\mu$, there exists some constant $C<\infty$ such that $\sup_{j\in\mathbb{N}}\int|u_j|^p\,d\mu\leqslant C$, and for every $\epsilon>0$ there is some $N_\epsilon\in\mathbb{N}$ such that
$$\left|\int|u_j|^p\,d\mu-\int|u|^p\,d\mu\right|\leqslant\epsilon^p\qquad\forall j\geqslant N_\epsilon.$$
Setting $w_\epsilon:=\max\{|u_1|,|u_2|,\dots,|u_{N_\epsilon}|,|u|\}$, we have $w_\epsilon\in\mathcal{L}^p_+(\mu)^{[\checkmark]}$ and we see for every $\epsilon\in(0,1)$ that
$$\left\{|u_j|>\tfrac{1}{\epsilon}\,w_\epsilon\right\}=\emptyset\quad\forall j\leqslant N_\epsilon,\qquad\left\{|u_j|>\tfrac{1}{\epsilon}\,w_\epsilon\right\}\subset\{\epsilon|u_j|>|u|\}\quad\forall j\in\mathbb{N}.$$
This implies for all $j\in\mathbb{N}$ that
$$\begin{aligned}\int_{\left\{|u_j|>\frac{1}{\epsilon}w_\epsilon\right\}}|u_j|^p\,d\mu&\leqslant\left|\int_{\left\{|u_j|>\frac{1}{\epsilon}w_\epsilon\right\}}(|u_j|^p-|u|^p)\,d\mu\right|+\int_{\left\{|u_j|>\frac{1}{\epsilon}w_\epsilon\right\}}|u|^p\,d\mu\\&\leqslant\epsilon^p+\int_{\{\epsilon|u_j|>|u|\}}|u|^p\,d\mu\\&\leqslant\epsilon^p+\epsilon^p\sup_{j\in\mathbb{N}}\int|u_j|^p\,d\mu\;\leqslant\;(1+C)\,\epsilon^p.\end{aligned}$$</td>
<td>(iii)⇒(ii): Fix $\epsilon\in(0,1)$. We have
$$\begin{aligned}\int\limits_{\{\epsilon|u_j|>|u|\}}|u_j|^p\,d\mu&\leqslant\int\limits_{\{\epsilon|u_j|>|u|\}}|u|^p\,d\mu+\int\limits_{\{\epsilon|u_j|>|u|\}}\big(|u_j|^p-|u|^p\big)\,d\mu\\&=\int\limits_{\{\epsilon|u_j|>|u|\}}|u|^p\,d\mu+\int\big(|u_j|^p-|u|^p\big)\,d\mu+\int\limits_{\{\epsilon|u_j|\leqslant|u|\}}\big(|u_j|^p-|u|^p\big)\,d\mu.\end{aligned}$$
Denote the three integrals I_1, I_2 and I_3, respectively. Because of assumption (iii) we know that
$$I_2\leqslant\epsilon^p,\quad\text{for all}\quad j\geqslant N_\epsilon$$
for some $N_\epsilon\in\mathbb{N}$. Moreover, (iii) shows that $\sup_{j\in\mathbb{N}}\int|u_j|\,d\mu\leqslant C<\infty$ which means that
$$I_1\leqslant\int\limits_{\{\epsilon|u_j|>|u|\}}|u|^p\,d\mu\leqslant\epsilon^p\int\limits_{\{\epsilon|u_j|>|u|\}}|u_j|^p\,d\mu\leqslant\epsilon^p\int|u_j|^p\,d\mu\leqslant C\,\epsilon^p.$$
Finally, the inclusion
$$\big\{\epsilon|u_j|\leqslant|u|\big\}\subset\big\{\big||u_j|^p-|u|^p\big|\leqslant(\epsilon^{-p}+1)|u|^p\big\}$$
yields with $\kappa:=\epsilon^{-p}+1$ and $\Delta_j^p:=\big||u_j|^p-|u|^p\big|$
$$|I_3|\leqslant\int_{\{\Delta_j^p\leqslant\kappa|u|^p\}}\Delta_j^p\,d\mu.$$</td>
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continuation		Splitting the integration domain into three parts, $$\big\{\Delta_j^p \leqslant (\kappa\|u\|^p)\wedge\eta\big\},\quad \big\{\eta\leqslant\Delta_j^p\leqslant\kappa\|u\|^p\big\}\cap\big\{\|u\|^p>R\big\},$$ $$\text{and}\quad \big\{\eta\leqslant\Delta_j^p\leqslant\kappa\|u\|^p\big\}\cap\big\{\|u\|^p\leqslant R\big\},$$ we get $$\|I_3\|\leqslant\int\eta\wedge(\kappa\|u\|^p)\,d\mu+\int_{\{\|u\|^p>R\}}\kappa\|u\|^p\,d\mu+\kappa\,R\,\mu\big(\big\{\Delta_j^p\geqslant\eta\big\}\cap\big\{\|u\|^p\geqslant\tfrac{\eta}{\kappa}\big\}\big).$$ By dominated convergence, the two integrals on the right hand side tend to 0 as $\eta\to0$ resp. $R\to\infty$. Thus, we may choose $\eta=\eta_\epsilon$ and $R=R_\epsilon$ such that $$\|I_3\|\leqslant\epsilon^p+\kappa\,R\,\mu\big(\big\{\Delta_j^p\geqslant\eta\big\}\cap\big\{\|u\|^p\geqslant\tfrac{\eta}{\kappa}\big\}\big).$$ Since the set $\{\|u\|^p\geqslant\eta/\kappa\}$ has finite μ-measure[✓] and since $\|u_j\|^p\xrightarrow{\mu}\|u\|^p$, i.e. $\Delta_j^p\xrightarrow{\mu}0$, see Lemma **??**(iii), we can find some $M_\epsilon\geqslant N_\epsilon$ such that $$\|I_3\|\leqslant2\epsilon^p\quad\text{for all}\quad j\geqslant M_\epsilon.$$ Setting $w_\epsilon:=\max\{\|u_1\|,\dots,\|u_{M_\epsilon}\|,\|u\|\}$ we have $w_\epsilon\in\mathcal{L}^p_+(\mu)$[✓] and see $$\sup_{j\in\mathbb{N}}\int_{\{\|u_j\|\geqslant\frac1\epsilon w_\epsilon\}}\|u_j\|\,d\mu\leqslant\sup_{j\geqslant M_\epsilon}\int_{\{\epsilon\|u_j\|\leqslant\|u\|\}}\|u_j\|\,d\mu\leqslant(C+3)\,\epsilon^p.$$

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PAGE, LINE	READS	SHOULD READ
p 167, lines 4–10 above	From this we conclude that for $W = 2w \in \mathcal{L}^p_+(\mu)$ and large $R > 0$ $\begin{aligned} \int \lvert u_j-u_k\rvert^p\,d\mu &= \int_{\{\lvert u_j-u_k\rvert>W\}} \lvert u_j-u_k\rvert^p\,d\mu + \int_{\{\lvert u_j-u_k\rvert\leqslant W\}} \lvert u_j-u_k\rvert^p\,d\mu \\ &\leqslant 2^{p+2}\,\epsilon + \int_{\{\lvert u_j-u_k\rvert\leqslant W\wedge\epsilon\}} \lvert u_j-u_k\rvert^p\,d\mu + \int_{\{\epsilon<\lvert u_j-u_k\rvert\leqslant W\}} \lvert u_j-u_k\rvert^p\,d\mu \\ &\leqslant 2^{p+2}\,\epsilon + \int \epsilon^p\wedge W^p\,d\mu + \Big\{ \int_{\substack{\{\lvert u_j-u_k\rvert>\epsilon\}\\ \cap\{W>R\}}} W^p\,d\mu + \int_{\substack{\{\lvert u_j-u_k\rvert>\epsilon\}\\ \cap\{\epsilon<W\leqslant R\}}} W^p\,d\mu \Big\} \\ &\leqslant 2^{p+2}\,\epsilon + \int \epsilon^p\wedge W^p\,d\mu + \int_{\{W>R\}} W^p\,d\mu \\ &\qquad + R^p\mu\big(\{\lvert u_j-u_k\rvert>\epsilon\}\cap\{\epsilon<W\leqslant R\}\big). \end{aligned}$ Letting first $j,k\to\infty$ we find because of $u_j \xrightarrow{\mu} u$ that[✓] $\limsup_{j,k\to\infty} \int \lvert u_j-u_k\rvert^p\,d\mu \leqslant 2^{p+2}\,\epsilon + \int \epsilon^p\wedge W^p\,d\mu + \int_{\{W>R\}} W^p\,d\mu.$ The last two terms vanish as $\epsilon\to 0$ and $R\to\infty$ by the dominated convergence theorem 12.9, so that $\lim_{j,k\to\infty}\int \lvert u_j-u_k\rvert^p\,d\mu = 0$. Since $\mathcal{L}^p(\mu)$ is complete (cf. T 12.7), $(u_j)_{j\in\mathbb{N}}$ converges in $\mathcal{L}^p(\mu)$ to a limit $\tilde u \in \mathcal{L}^p(\mu)$.	From this we conclude that for $W = 2w \in \mathcal{L}^p_+(\mu)$ and large $R > 0$ $\begin{aligned} \int \lvert u_j-u_k\rvert^p\,d\mu &= \int_{\{\lvert u_j-u_k\rvert>W\}} \lvert u_j-u_k\rvert^p\,d\mu + \int_{\{\lvert u_j-u_k\rvert\leqslant W\}} \lvert u_j-u_k\rvert^p\,d\mu \\ &\leqslant 2^{p+2}\,\epsilon + \int_{\{\lvert u_j-u_k\rvert\leqslant W\wedge\delta\}} \lvert u_j-u_k\rvert^p\,d\mu + \int_{\{\delta<\lvert u_j-u_k\rvert\leqslant W\}} \lvert u_j-u_k\rvert^p\,d\mu \\ &\leqslant 2^{p+2}\,\epsilon + \int \delta^p\wedge W^p\,d\mu + \Big\{ \int_{\substack{\{\lvert u_j-u_k\rvert>\delta\}\\ \cap\{W>R\}}} W^p\,d\mu + \int_{\substack{\{\lvert u_j-u_k\rvert>\delta\}\\ \cap\{\delta<W\leqslant R\}}} W^p\,d\mu \Big\} \\ &\leqslant 2^{p+2}\,\epsilon + \int \delta^p\wedge W^p\,d\mu + \int_{\{W>R\}} W^p\,d\mu \\ &\qquad + R^p\mu\big(\{\lvert u_j-u_k\rvert>\delta\}\cap\{\delta<W\leqslant R\}\big). \end{aligned}$ Letting first $j,k\to\infty$ we find because of $u_j \xrightarrow{\mu} u$ that[✓] $\limsup_{j,k\to\infty} \int \lvert u_j-u_k\rvert^p\,d\mu \leqslant 2^{p+2}\,\epsilon + \int \delta^p\wedge W^p\,d\mu + \int_{\{W>R\}} W^p\,d\mu.$ The last two terms vanish as $\delta\to 0$ and $R\to\infty$ by the dominated convergence theorem 12.9, so that $\lim_{j,k\to\infty}\int \lvert u_j-u_k\rvert^p\,d\mu = 0$. Since $\mathcal{L}^p(\mu)$ is complete (cf. T 12.7), $(u_j)_{j\in\mathbb{N}}$ converges in $\mathcal{L}^p(\mu)$ to a limit $\tilde u \in \mathcal{L}^p(\mu)$.
p 173, Prob. 16.1, line 20 below	$(X, \mathcal{A}, \nu)$	$(X, \mathcal{A}, \mu)$
p 173, Prob. 16.1, line 19 below	$f_j \xrightarrow{j\to\infty} 0$	$u_j \xrightarrow{j\to\infty} 0$
p 173, Prob. 16.3, line 13 below	be a measure space	be a σ-finite measure space
p 174, Prob. 16.8	ρ_μ, g_μ, d_μ *etc.*	ρ_P, g_P, d_P *etc.*
p 177, line 8 below	Therefore (ii) means	Therefore (i) means
p 188, Prob. 17.1	$u_0 = \int u_1\,d\mu$	$u_0 = \mu(X)^{-1}\int u_1\,d\mu$
p 188, Prob. 17.7	Set $f_0 := 0$ and $u_0 := \int u_1\,d\mu$.	Set $f_0 := 0$.
p 191, line 11 above	$(b-a)U([a,b];N)$	$(b-a)U([a,b];N) - (X_N - a)^-$
p 191, line 14 above	$(b-a)\int_A U([a,b];N)\,d\mu$	$(b-a)\int_A U([a,b];N)\,d\mu - \int_A (X_N-a)^-\,d\mu$

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p 191, line 8 below	$\overset{17.7}{\leqslant} \int_A (u_{\tau_N} - a)\,d\mu \leqslant \int_A (u_{\tau_N} - a)^+\,d\mu$	$\overset{17.7}{\leqslant} \int_A (u_{\tau_N} - a)\,d\mu$
p 200, Prob. 18.6(ii)	Let $\mathcal{A}_n = \sigma(X_1, \dots, X_{2^n})$. Show that $A_n := \{X_{2^{n-1}+1} + \dots + X_{2^n} = 0\}$ is for each $n \in \mathbb{N}$ contained in $\mathcal{A}_n$ and	Let $\mathcal{A}_n = \sigma(X_1, \dots, X_{n^2})$. Show that $A_n := \{X_{(n-1)^2+2} + \dots + X_{n^2} = 0\}$ is for each $n \in \mathbb{N}, n \geqslant 2$ contained in $\mathcal{A}_n$ and
p 200, Prob. 18.6(iii)	The sequence $M_0 := 0$ and $M_{n+1} := M_n(1 + X_{2^n+1}) + \mathbf{1}_{A_n} X_{2^n+1}$, $n \in \mathbb{N}_0$, defines a martingale $(M_n, \mathcal{A}_n)_{n \geqslant 1}$.	The sequence $M_2 := 0$ and $M_{n+1} := M_n(1 + X_{n^2+1}) + \mathbf{1}_{A_n} X_{n^2+1}$, $n \geqslant 2$, defines a martingale $(M_n, \mathcal{A}_n)_{n \geqslant 2}$.
p 214, line 2 above	$\sup\left\{\frac{1}{\lambda^n(Q)} \int_Q \lvert f\rvert\,d\lambda^n \,:\, Q \in \bigcup_{k \in \mathbb{Z}} \mathcal{A}_k^{[0]},\ x \in Q\right\}$	$\sup\left\{\frac{1}{\lambda^n(Q)} \int_Q \lvert f\rvert\,d\lambda^n \,:\, Q = Q_k(z),\ k \in \mathbb{Z},\ z \in 2^{-k}\mathbb{Z}^n,\ x \in Q\right\}$
p 217, line 6 below	$\sup\left\{\frac{\mu(Q)}{\lambda^n(Q)} \,:\, Q \in \bigcup_{k \in \mathbb{Z}} \mathcal{A}_k^{[\mathbf{e}]},\ x \in Q\right\}$	$\sup\left\{\frac{\mu(Q)}{\lambda^n(Q)} \,:\, Q = Q_k(z),\ k \in \mathbb{Z},\ z \in 2^{-k}\mathbb{Z}^n,\ x \in Q\right\}$
p 223, Prob. 19.9	$(X, \mathcal{A}, \mu)$	$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$
p 223, Prob. 19.9	**drop the hint**	
p 225, Prob. 19.14	A martingale $(u_j, \mathcal{A}_j)_{j \in \mathbb{N}}$ is called	A martingale $(u_j, \mathcal{A}_j)_{j \in \mathbb{N}}$ on a σ-finite filtered measure space is called
p 231, (20.13)	$\dots + i\lVert\lVert v - iw\rVert^2 - i\lVert v - iw\rVert^2$	$\dots + i\lVert\lVert v + iw\rVert^2 - i\lVert v + iw\rVert^2$
p 233, Prob. 20.5	an inner product space	a real inner product space
p 237, line 3 below	$\dots, \underbrace{\alpha g + \beta h}_{\in F}\rangle = 0$	$\dots, \underbrace{\alpha P_F g + \beta P_F h}_{\in F}\rangle = 0$
p 237, line 1 below	$\dots, \underbrace{\alpha g + \beta h}_{\in F}\rangle = 0$	$\dots, \underbrace{P_F(\alpha g + \beta h)}_{\in F}\rangle = 0$
p 247, Prob. 21.13 (iv)	Show that $\left(E^{\mathcal{A}_n} u\right)_{n \in \mathbb{N} \cup \{\infty\}}$ is a martingale.	Show that $\left(E^{\mathcal{A}_n} u\right)_{n \in \mathbb{N} \cup \{\infty\}}$, $u \in L^1(\mathcal{A}) \cap L^2(\mathcal{A})$, is a martingale.
p 247, Prob. 21.13 (v)	Show that $E^{\mathcal{A}_n} u \xrightarrow{n \to \infty} E^{\mathcal{A}_\infty} u$ a.e. and in L^2.	Show that $E^{\mathcal{A}_n} u \xrightarrow{n \to \infty} E^{\mathcal{A}_\infty} u$ a.e. and in L^2 for all $u \in L^1(\mathcal{A}) \cap L^2(\mathcal{A})$.
p 261, line 6 above	$\mathbf{1}_{\{0 < \lvert E^{\mathcal{G}} u\rvert < \infty\}}$	$\mathbf{1}_{\{0 < \lvert \mathsf{E}^{\mathcal{G}} u\rvert < \infty\}}$
p 261, line 14 below	$\left\lVert \mathsf{E}^{\mathcal{G}} u\, \mathbf{1}_{G_n}\right\rVert_1 = \left\langle \mathsf{E}^{\mathcal{G}} u, \mathbf{1}_{G_n}\right\rangle \overset{21.4(\text{iii}),(\text{ix})}{=} \left\langle u, \mathbf{1}_{G_n}\right\rangle \leqslant \lVert u\rVert_1$	$\left\lVert \mathsf{E}^{\mathcal{G}} u\, \mathbf{1}_{G_n}\right\rVert_1 \overset{22.4(\text{xii})}{\leqslant} \left\langle \mathsf{E}^{\mathcal{G}} \lvert u\rvert, \mathbf{1}_{G_n}\right\rangle \overset{22.4(\text{iii}),(\text{ix})}{=} \left\langle \lvert u\rvert, \mathbf{1}_{G_n}\right\rangle \leqslant \lVert u\rVert_1$
p 274, problem 23.4	$M(\mathcal{A})$	$M^+(\mathcal{A})$
p 274, problem 23.6	$L^1(\mathcal{A})$	$L^1_+(\mathcal{A})$
p 275, problem 23.11	**twice:** s_j	**twice:** u_j
p 281, lines 10–11 above	$C[0,1] \subset L^1[0,1] \subset L^2[0,1]$	$C[0,1] \subset L^2[0,1] \subset L^1[0,1]$

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PAGE, LINE	READS	SHOULD READ
p 283, (24.4)	... from the classical result that $\displaystyle\int_{(-\pi,\pi)} \cos kx\,\sin \ell x\,dx = \begin{cases} 0, & \text{if } k \neq \ell, \\ \pi, & \text{if } k = \ell \geqslant 1, \\ 2\pi, & \text{if } k = \ell = 0, \end{cases}$ (24.4)	... from the classical result that for $j,k \in \mathbb{N}_0$ and $\ell, m \in \mathbb{N}$ $\displaystyle\int_{(-\pi,\pi)} \cos jx\,\cos kx\,dx = \begin{cases} 0, & \text{if } k \neq j, \\ \pi, & \text{if } k = j \geqslant 1, \\ 2\pi, & \text{if } k = j = 0, \end{cases}$ $\displaystyle\int_{(-\pi,\pi)} \sin \ell x\,\sin mx\,dx = \begin{cases} 0, & \text{if } \ell \neq m, \\ \pi, & \text{if } \ell = m \end{cases}$ (24.4) $\displaystyle\int_{(-\pi,\pi)} \cos kx\,\sin \ell x\,dx = 0 \quad \text{for all} \quad k, \ell$
p 286, before Lemma 24.11	**add the following lime just before Lemma 24.11**	Denote the set of 2π-periodic continuous functions by $C_{\text{per}}[-\pi,\pi] = \big\{u \in C[-\pi,\pi] : u(\pi) = u(-\pi)\big\}$
p 286, line 6 below	$C[-\pi,\pi]$	$C_{\text{per}}[-\pi,\pi]$
p 287, line 15, 13, 12, 11 below	$C[-\pi,\pi]$	$C_{\text{per}}[-\pi,\pi]$
p 297, line 9 below	$\big(\mathsf{E}^{\mathcal{A}^{\Delta}_{-M}}\big)_{M\in\mathbb{N}}$	$\big(\mathsf{E}^{\mathcal{A}^{\Delta}_{-M}} u\big)_{M\in\mathbb{N}}$
p 309, line 5 below	most prominent stochastic process	most prominent stochastic processes
p 310, line 3 above	$B \in \mathcal{B}[0,1]$	$B \in \mathcal{B}(\mathbb{R})$
p 310, lines 9-12 above	$\displaystyle\langle \mathbf{1}_{[0,t]}, H_n\rangle = \int_0^t H_n(x)\,dx = \tfrac12\, 2^{k/2} \int_0^t H_1(2^k x - j)\,dx = \tfrac12\, 2^{-k/2}\, F_n(t),$ where $F_1(t) = \int_0^t H_1(x)\,dx\,\mathbf{1}_{[0,1]}(t) = 2t\mathbf{1}_{[0,\frac12)}(t) - (2t-2)\mathbf{1}_{[\frac12,1]}(t)$ is a tent-function and $F_n(t) := F_1(2^k t - j)$. Since $0 \leqslant F_n \leqslant 1$, we see $\displaystyle\sum_{n=0}^{\infty} \lvert\langle \mathbf{1}_{[0,t]}, H_n\rangle\rvert^2 \leqslant \frac14 \sum_{n=0}^{\infty} 2^{-k} = \frac12\,,$	$\displaystyle\langle \mathbf{1}_{[0,t]}, H_n\rangle = \int_0^t H_n(x)\,dx = 2^{k/2} \int_0^t H_1(2^k x - j)\,dx = 2^{-k/2}\, F_n(t),$ where $F_1(t) = \int_0^t H_1(x)\,dx\,\mathbf{1}_{[0,1]}(t) = t\mathbf{1}_{[0,\frac12)}(t) - (t-1)\mathbf{1}_{[\frac12,1]}(t)$ is a tent-function and $F_n(t) := F_1(2^k t - j)$. Since $0 \leqslant F_n \leqslant \frac12$, we see $\displaystyle\sum_{n=1}^{\infty} \lvert\langle \mathbf{1}_{[0,t]}, H_n\rangle\rvert^2 \leqslant \frac14 \sum_{n=1}^{\infty} 2^{-n} = \frac14\,,$
p 310, footnote	$e^{\sigma_n^2 \xi^2/2}$ (twice) resp. $e^{\sigma^2 \xi^2/2}$	$e^{-\sigma_n^2 \xi^2/2}$ (twice) resp. $e^{-\sigma^2 \xi^2/2}$

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PAGE, LINE	READS	SHOULD READ
p 311, line 10-13 below	**the paragraph (line 14 below − 6 below):** "Let us finally turn ... a.e. limit $W_t(\omega)$" **is corrupted (the exponent** 4 **is missing) and should be corrected as follows (see also bonus material for an extended version):**	Let us finally turn to the dependence of $W_t(\omega)$ on t. Note that for $2^m-1=M<N=2^n-1$ $$\int_0^1 \sup_{t\in[0,1]} \big\|S_N(t;\omega)-S_M(t;\omega)\big\|^4\,d\omega = \int_0^1 \sup_{t\in[0,1]} \Big\|\sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]},H_{2^k+j}\rangle\Big\|^4 d\omega = \int_0^1 \sup_{t\in[0,1]} \Big\|\sum_{k=m}^{n-1} 2^{-\frac{k}{8}}\Big[\sum_{j=0}^{2^k-1} 2^{\frac{k}{8}} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]},H_{2^k+j}\rangle\Big]\Big\|^4 d\omega \leqslant \int_0^1 \sup_{t\in[0,1]} \underbrace{\Big[\sum_{k=m}^{n-1}2^{-\frac{k}{6}}\Big]}_{\leqslant 10}\cdot\sum_{k=m+1}^{n}\Big[\sum_{j=0}^{2^k-1}2^{\frac{k}{8}} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]},H_{2^k+j}\rangle\Big]^4 d\omega$$ where we used Hölder's inequality for the outer sum with $p=\frac{4}{3}$ and $q=4$. Since the functions $t\mapsto\langle \mathbf{1}_{[0,1]},H_{2^k+j}\rangle$ have for $j=0,\dots,2^k-1$ disjoint supports and are bounded by $\frac{1}{2}2^{-k/2}$, we find $$\int_0^1 \sup_{t\in[0,1]} \big\|S_N(t;\omega)-S_M(t;\omega)\big\|^4\,d\omega \leqslant 10\int_0^1 \sup_{t\in[0,1]} \sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} 2^{\frac{k}{2}} e^4_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]},H_{2^k+j}\rangle^4\,d\omega \leqslant 10\sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} 2^{\frac{k}{2}}\underbrace{\int_0^1 e^4_{2^k+j}(\omega)\,d\omega}_{=C\ \ \forall j,k}\,2^{4-2k} = 10C\sum_{k=m}^{n-1}2^{\frac{k}{2}}\cdot 2^k\cdot 2^{4-2k} \leqslant \frac{5C}{4}\,2^{-\frac{m}{2}}$$ which means that the partial sums $S_N(t;\omega)$ of $W_t(\omega)$ converge in $L^1(d\omega)$ uniformly for all $t\in[0,1]$. By C**??** we can extract a subsequence, which converges (uniformly in t) for $\lambda(d\omega)$-almost all ω to $W_t(\omega)$; since for fixed ω the partial sums $t\mapsto S_N(t;\omega)$ are continuous functions of t, this property is inherited by the a.e. limit $W_t(\omega)$.

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PAGE, LINE	READS	SHOULD READ
p 312, Prob. 24.5	$\sin^{k+1} x$	$\sin^{2k+1} x$
p 313, line 9 above	$[-\infty, +\infty)$ and $(-\infty, +\infty]$, respectively.	$[-\infty, +\infty]$.
p 314, line 9 above	**(ix)** If, for all $j \in \mathbb{N}$, $a_j, b_j \geqslant 0$, then	**(ix)** For all bounded sequences $a_j, b_j \geqslant 0$, $j \in \mathbb{N}$, we have
p 314, line 14 above	**(xi)** If $a_j, b_j \geqslant 0$ for all $j \in \mathbb{N}$ and if $\lim_{j\to\infty} a_j$ exists, then	**(xi)** If $a_j, b_j \geqslant 0$ for all $j \in \mathbb{N}$ such that $(b_j)_{j\in\mathbb{N}}$ is bounded and $\lim_{j\to\infty} a_j$ exists, then
p 320, Prop. B.6	**(ii)** Closed subsets of compact sets are closed.	**(ii)** Closed subsets of compact sets are compact.
p 365, Ref. [22]	Kaczmarz, S. and H. Steinhaus, *Theorie der Orthogonalreihen* (2nd corr. reprint), New York: Chelsea, 1951. First edition appeared under the same title with PWN, Warsaw: Monogr. Mat. Warszawa vol. VI, 1935.	Kaczmarz, S. and H. Steinhaus, *Theorie der Orthogonalreihen* (2nd corr. reprint), New York: Chelsea, 1951. First edition was published in Warsaw: PWN, Monogr. Mat. Warszawa vol. **VI**, 1935.
p 365, Ref. [34]	Paley, R. E. A. C. and N. Wiener, Providence (RI): *Fourier Transforms in the Complex Domain*, American Mathematical Society, Coll. Publ. vol. **19**, 1934.	Paley, R. E. A. C. and N. Wiener, *Fourier Transforms in the Complex Domain*, Providence (RI): American Mathematical Society, Coll. Publ. vol. **19**, 1934.
p 372, index item *completion and inner/outer regularity*	160, (Pr 15.6)	160, (Pr 15.3)

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