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Solution Manual Chapter 13–24

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13 Product measures and Fubini's theorem

Solutions to Problems 13.1–13.14

Problem 13.1

- We have

$$(x,y) \in \left(\bigcup_i A_i\right) \times B \iff x \in \bigcup_i A_i \text{ and } y \in B$$
$$\iff \exists\, i_0 : x \in A_{i_0} \text{ and } y \in B$$
$$\iff \exists\, i_0 : (x,y) \in A_{i_0} \times B$$
$$\iff (x,y) \in \bigcup_i (A_i \times B).$$

- We have

$$(x,y) \in \left(\bigcap_i A_i\right) \times B \iff x \in \bigcap_i A_i \text{ and } y \in B$$
$$\iff \forall\, i : x \in A_i \text{ and } y \in B$$
$$\iff \forall\, i : (x,y) \in A_i \times B$$
$$\iff (x,y) \in \bigcap_i (A_i \times B).$$

- Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 120 and the fact that inverse maps interchange with all set operations, we get

$$\begin{aligned}(A \times B) \cap (A' \times B') &= \left[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\right] \cap \left[\pi_1^{-1}(A') \cap \pi_2^{-1}(B')\right] \\ &= \left[\pi_1^{-1}(A) \cap \pi_1^{-1}(A')\right] \cap \left[\pi_2^{-1}(B) \cap \pi_2^{-1}(B')\right] \\ &= \pi_1^{-1}(A \cap A') \cap \pi_2^{-1}(B \cap B') \\ &= (A \cap A') \times (B \cap B').\end{aligned}$$

- Using the formula $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ (see page 120 and the fact that inverse maps interchange with all set operations, we get

$$\begin{aligned}A^c \times B &= \pi_1^{-1}(A^c) \cap \pi_2^{-1}(B) \\ &= \left[\pi_1^{-1}(A)\right]^c \cap \pi_2^{-1}(B) \\ &= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \left[\pi_1^{-1}(A)\right]^c\end{aligned}$$

$$\begin{aligned}
&= \pi_1^{-1}(X) \cap \pi_2^{-1}(B) \cap \left\{ \left[\pi_1^{-1}(A)\right]^c \cup \left[\pi_2^{-1}(B)\right]^c \right\} \\
&= (X \times B) \cap \left[\pi_1^{-1}(A) \cap \pi_2^{-1}(B)\right]^c \\
&= (X \times B) \cap \left[A \times B\right]^c \\
&= (X \times B) \setminus (A \times B).
\end{aligned}$$

- We have

$$\begin{aligned}
A \times B \subset A' \times B' &\iff \left[(x,y) \in A \times B \implies (x,y) \in A' \times B'\right] \\
&\iff \left[x \in A, y \in B \implies x \in A', y \in B'\right] \\
&\iff A \subset A', \;\; B \subset B'.
\end{aligned}$$

Problem 13.2 Pick two exhausting sequences $(A_k)_k \subset \mathcal{A}$ and $(B_k)_k \subset \mathcal{B}$ such that we have $\mu(A_k), \nu(B_k) < \infty$ and $A_k \uparrow X$, $B_k \uparrow Y$. Then, because of the continuity of measures,

$$\begin{aligned}
\mu \times \nu(A \times N) &= \lim_k \mu \times \nu\big((A \times N) \cap (A_k \times B_k)\big) \\
&= \lim_k \mu \times \nu\big((A \cap A_k) \times (N \cap B_k)\big) \\
&= \lim_k \Big[\underbrace{\mu(A \cap A_k)}_{<\infty} \cdot \underbrace{\nu(N \cap B_k)}_{\leqslant \nu(N) = 0} \Big] \\
&= 0.
\end{aligned}$$

Since $A \times N \in \mathcal{A} \times \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B}$, measurability is clear.

Problem 13.3 Since the two expressions are symmetric in x and y, they must coincide if they converge. Let us, therefore only look at the left hand side.

The inner integral,

$$\int_{(0,\infty)} e^{-xy} \sin x \, \lambda(dx)$$

clearly satisfies

$$\begin{aligned}
\int_{(0,\infty)} \left|e^{-xy} \sin x\right| \lambda(dx) &\leqslant \int_{(0,\infty)} e^{-xy} \, \lambda(dx) \\
&= \int_0^\infty e^{-xy} \, dx \\
&= \left[-\frac{e^{-xy}}{y} \right]_{x=0}^{\infty} \\
&= \frac{1}{x}.
\end{aligned}$$

Since the integrand is continuous and has only one sign, we can use Riemann's integral. Thus, the integral exists. To calculate its value we observe that two integrations by parts yield

$$\int_0^\infty e^{-xy} \, sinx \, dx = -e^{-xy} \cos x \Big|_{x=0}^{\infty} - \int_0^\infty y e^{-xy} \cos x \, dx$$

$$
\begin{aligned}
&= 1 - y \int_0^\infty e^{-xy} \cos x \, dx \\
&= 1 - y\left(e^{-xy} \sin x \Big|_{x=0}^{\infty} + \int_0^\infty y e^{-xy} \sin x \, dx \right) \\
&= 1 - y^2 \int_0^\infty e^{-xy} \sin x \, dx.
\end{aligned}
$$

And if we solve this equality for the integral expression, we get

$$
(1 + y^2) \int_0^\infty e^{-xy} \sin x \, dx = 1 \implies \int_0^\infty e^{-xy} \sin x \, dx = \frac{1}{1 + y^2}.
$$

Alternative: Since $\sin x = \operatorname{Im} e^{ix}$ we get

$$
\int_0^\infty e^{-xy} \, sinx \, dx = \operatorname{Im} \int_0^\infty e^{-(y-i)x} \, dx = \operatorname{Im} \frac{1}{y - i} = \operatorname{Im} \frac{y + i}{y^2 + 1} = \frac{1}{y^2 + 1}.
$$

Thus the iterated integral exists, since

$$
\int_{(0,\infty)} \left| \frac{\sin x}{1 + x^2} \right| dx \leqslant \int_{(0,\infty)} \frac{1}{1 + x^2} \, dx = \arctan x \Big|_0^\infty = \frac{\pi}{2}.
$$

(Here we used again that improper Riemann integrals with positive integrands coincide with Lebesgue integrals.)

In principle, the existence and equality of iterated integrals is not good enough to guarantee the existence of the double integral. For this one needs the existence of the *absolute* iterated integrals—cf. Tonelli's theorem 13.8. In the present case one can see that the absolute iterated integrals exist, though:

On the one hand we find

$$
\int_{(0,\infty)} e^{-xy} |\sin(x)| \, \lambda(dx) \leqslant \frac{e^{-xy}}{-y} \Big|_0^\infty = \frac{1}{y}
$$

and $\frac{\sin y}{y}$ is, as a bounded continuous function, Lebesgue integrable over $(0, 1)$.

On the other hand we can use integration by parts to get

$$
\begin{aligned}
\int_{k\pi}^{(k+1)\pi} e^{-xy} \sin x \, dx &= \frac{e^{-xy}}{-y} \sin x \Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi} \frac{e^{-xy}}{-y} \cos x \, dx \\
&= \frac{e^{-xy}}{-y^2} \cos x \Big|_{k\pi}^{(k+1)\pi} - \int_{k\pi}^{(k+1)\pi} \frac{e^{-xy}}{-y^2} (-1) \sin x \, dx
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\frac{y^2 + 1}{y^2} \int_{k\pi}^{(k+1)\pi} e^{-xy} \sin x \, dx &= \frac{e^{-(k+1)\pi y}}{-y^2} (-1)^{k+1} - \frac{e^{-k\pi y}}{-y^2} (-1)^k \\
&= \frac{(-1)^k}{y^2} \left(e^{-(k+1)\pi y} + e^{-k\pi y} \right),
\end{aligned}
$$

i.e. $\int_{k\pi}^{(k+1)\pi} e^{-xy} \sin x \; dx = (-1)^k \frac{1}{y^2+1} \left(e^{-(k+1)\pi y} + e^{-k\pi y} \right)$.

Now we find a bound for $y \in (1,\infty)$.

$$
\begin{aligned}
\int_{(0,\infty)} e^{-xy}|\sin(x)|dx &= \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} e^{-xy} \sin x(-1)^k \; dx \\
&= \sum_{k=0}^{\infty} (-1)^k(-1)^k \frac{1}{y^2+1}(e^{-(k+1)\pi y} + e^{-k\pi y}) \\
&\leqslant \frac{2}{y^2+1} \sum_{k=0}^{\infty} (e^{-\pi y})^k \\
&\overset{y>1}{\leqslant} \frac{2}{y^2+1} \sum_{k=0}^{\infty} (e^{-\pi})^k
\end{aligned}
$$

which means that the left hand side is integrable over $(1,\infty)$.

Thus we have

$$
\begin{aligned}
&\int_{(0,\infty)} \int_{(0,\infty)} |e^{-xy} \sin x \sin y| \,\lambda(dx)\,\lambda(dy) \\
&\leqslant \int_{(0,1]} \frac{\sin y}{y}\,\lambda(dy) + \int_{(1,\infty)} \frac{2}{y^2+1}\,\lambda(dy) \sum_{k=0}^{\infty} (e^{-\pi})^k \\
&< \infty.
\end{aligned}
$$

By Fubini's theorem we know that the iterated integrals as well as the double integral exist and their values are identical.

The following much shorter and more elegant proof for the absolute convergence of the integral I learned from Alvaro H. Salas (Universidad Nacional de Colombia, Department of Mathematics, July 2012):

Let

$$
f(x,y) = e^{-xy}\,|\sin x\, \sin y| \geqslant 0 \quad \forall x,y \geqslant 0.
$$

By monotone convergence and Tonelli's theorem

$$
\begin{aligned}
\iint f(x,y)\,dx\,dy &= \lim_{A,B\to\infty} \iint_{(0,A]\times(0,B]} f(x,y)dx\,dy \\
&= \sup_{A,B\geqslant 0} \int_{(0,A]} \int_{(0,B]} f(x,y)\,dy\,dx.
\end{aligned}
$$

Since the integrands are bounded and continuous, we can use Riemann integrals. Fix $A > 1$ and $B > 1$. Then

$$
\int_0^A \int_0^B = \int_0^1 \int_0^1 + \int_0^1 \int_1^B + \int_0^1 \int_1^A + \int_1^A \int_1^B
$$

Now we can estimate these expressions separately: since $|\sin t| \leqslant |t|$ we have

$$
\begin{aligned}
\int_0^1 \int_0^1 f(x,y)\,dy\,dx &\leqslant \int_0^1 \int_0^1 1\,dx\,dy = 1. \\
\int_0^1 \int_1^B f(x,y)\,dy\,dx &\leqslant \int_1^B \left[\int_0^1 xe^{-xy}\,dx\right] dy \\
&= 1 - \frac{1}{e} + \frac{e^{-B}-1}{B} < 1 - \frac{1}{e}.
\end{aligned}
$$

$$\int_0^1 \int_1^A f(x,y)\,dx\,dy \leqslant \int_1^A \left[\int_0^1 ye^{-xy}\,dy\right] dx$$
$$= 1 - \frac{1}{e} + \frac{e^{-A}-1}{A} < 1 - \frac{1}{e}.$$
$$\int_1^A \int_1^B f(x,y)\,dx\,dy \leqslant \int_1^B \left[\int_1^A xe^{-xy}\,dx\right] dy$$
$$= \frac{1}{e} - e^{-A} + \frac{e^{-AB}-e^{-B}}{B} < \frac{1}{e}.$$

These estimates now show

$$\int_0^\infty \int_0^\infty e^{-xy}\,|\sin x\,\sin y|\,dx\,dy \leqslant 3 - \frac{1}{e}.$$

Problem 13.4 Note that

$$\frac{d}{dy}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}\,.$$

Thus we can compute

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \int_{(0,1)} \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

By symmetry of x and y in the integrals it follows that

$$\int_{(0,1)}\int_{(0,1)} \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx = -\frac{\pi}{4}$$

and therefore the double integral can not exist. Since the existence would imply the equality of the two above integrals. We can see this directly by

$$\int_{(0,1)}\int_{(0,1)} \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\,dx \geqslant \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx$$
$$= \int_0^1 \frac{x}{x^2+x^2}\,dx$$
$$= \frac{1}{2}\int_0^1 \frac{1}{x}\,dx = \infty.$$

Problem 13.5 Since the integrand is odd, we have for $y \neq 0$:

$$\int_{(-1,1)} \frac{xy}{(x^2+y^2)^2}\,dx = 0$$

and $\{0\}$ is a null set. Thus the iterated integrals have common value 0. But the double integral does not exist, since for the iterated absolute integrals we get

$$\int_{(-1,1)} \left|\frac{xy}{(x^2+y^2)^2}\right| dx = \frac{1}{|y|}\int_0^{1/|y|} \frac{\xi}{(\xi^2+1)^2}\,d\xi \qquad \geqslant \frac{2}{|y|}\underbrace{\int_0^1 \frac{\xi}{(\xi^2+1)^2}\,d\xi}_{<\infty}.$$

Here we used the substitution $x = \xi|y|$ and the fact that $|y| \leqslant 1$, thus $1/|y| \geqslant 1$. But the outer integral is bounded below by

$$\int_{(-1,1)} \frac{2}{|y|}\,dy \quad \text{which is divergent.}$$

Problem 13.6 (i) Since for continuous integrands over a compact interval the Riemann and Lebesgue integrals coincide, we find

$$\begin{aligned}\lim_{k\to\infty}\int_{(0,k)} e^{-tx}\,\lambda(dt) &= \lim_{k\to\infty}\int_{[0,k]} e^{-tx}\,dt\\ &= \lim_{k\to\infty}\frac{e^{-tx}}{-x}\Big|_0^k\\ &= \lim_{n\to\infty}\frac{e^{-kx}}{-x}-\frac{1}{-x}=\frac{1}{x}.\end{aligned}$$

(ii) Since $|\sin x\int_{(0,k)} e^{-tx}\,dt| \leqslant |\frac{\sin x}{x}|$ and since $\sin x/x$ is continuous and bounded on the interval $[0,n]$, we can use dominated convergence to get

$$\begin{aligned}\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx) &= \lim_{n\to\infty}\int_{(0,n)}\sin x\lim_{k\to\infty}\int_{(0,k)} e^{-tx}\,dt\,dx\\ &= \lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,n)}\int_{(0,k)}\sin x\,e^{-tx}\,dt\,dx.\end{aligned}$$

Since the integrand is continuous and since we integrate over a (relatively) compact set we can use Fubini's theorem and find

$$\begin{aligned}&\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx)\\ &\quad=\lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,k)}\int_{(0,n)}\sin x\,e^{-tx}\,dx\,dt\\ &\quad=\lim_{n\to\infty}\lim_{k\to\infty}\int_{(0,k)}\frac{1}{t^2+1}\Big(1-e^{-nt}(\cos n+t\sin n)\Big)\,dt\end{aligned}$$

where we also used that

$$\int_a^b e^{-xy}\sin x\,dx=\frac{1}{y^2+1}\big(e^{-ay}(\cos a+y\sin a)-e^{-by}(\cos b+y\sin b)\big)$$

Since

$$\begin{aligned}&\left|\mathbb{1}_{(0,k)}(t)\frac{1}{t^2+1}\big(1-e^{-nt}(\cos n+t\sin n)\big)\right|\\ &\qquad\leqslant\frac{2}{t^2+1}+\frac{t}{t^2+1}e^{-nt}\in L^1(0,\infty),\end{aligned}$$

dominated convergence yields

$$\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx)=\lim_{n\to\infty}\int_{(0,\infty)}\frac{1}{t^2+1}\Big(1-e^{-nt}(\cos n+t\sin n)\Big)\,dt$$

and, again by dominated convergence, since the integrand is for $n>1$ bounded by the integrable function $(0,\infty)\ni t\mapsto\frac{2}{t^2+1}+\frac{t}{t^2+1}\,e^{-t}$

$$\lim_{n\to\infty}\int_{(0,n)}\frac{\sin x}{x}\,\lambda(dx)=\int_{(0,\infty)}\frac{1}{t^2+1}\,dt=\arctan t\Big|_0^\infty=\frac{\pi}{2}.$$

Problem 13.7 Note that the diagonal $\Delta\subset\mathbb{R}^2$ is measurable, i.e. the (double) integrals are well-defined. The inner integral on the l.h.S. satisfies

$$\int_{[0,1]}\mathbb{1}_\Delta(x,y)\,\lambda(dx)=\lambda(\{y\})=0\qquad\forall\,y\in[0,1]$$

so that the left-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\lambda(dx)\,\mu(dy) = \int_{[0,1]} 0\,\mu(dy) = 0.$$

On the other hand, the inner integral on the right-hand side equals

$$\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\mu(dy) = \mu(\{x\}) = 1 \quad \forall\, x \in [0,1]$$

so that the right-hand side

$$\int_{[0,1]}\int_{[0,1]} \mathbb{1}_\Delta(x,y)\,\mu(dy)\,\lambda(dx) = \int_{[0,1]} 1\,\lambda(dx) = 1.$$

This shows that the double integrals are not equal. This does not contradict Tonelli's theorem since μ is not σ-finite.

Problem 13.8 (i) Note that, due to the countability of $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ there are no problems with measurability and σ-finiteness (of the counting measure).

Tonelli's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}}$ be a double sequence of positive numbers $a_{jk} \geqslant 0$. Then

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{jk} = \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{jk}$$

with the understanding that both sides are either finite or infinite.

Fubini's Theorem. Let $(a_{jk})_{j,k\in\mathbb{N}} \subset \mathbb{R}$ be a double sequence of real numbers a_{jk}. If

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} |a_{jk}| \quad\text{or}\quad \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} |a_{jk}|$$

is finite, then all of the following expressions converge absolutely and sum to the same value:

$$\sum_{j\in\mathbb{N}}\Bigg(\sum_{k\in\mathbb{N}} |a_{jk}|\Bigg), \quad \sum_{k\in\mathbb{N}}\Bigg(\sum_{j\in\mathbb{N}} |a_{jk}|\Bigg), \quad \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} |a_{jk}|.$$

(ii) Consider the (obviously σ-finite) measures $\mu_j := \sum_{k\in A_j}\delta_k$ and $\nu = \sum_{j\in\mathbb{N}}\mu_j$. Tonelli's theorem tells us that

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\sum_{k\in A_j} |x_k| &= \int_{\mathbb{N}}\int_{\mathbb{N}} |x_k|\,\mu_j(dk)\,\mu(dj)\\
&= \int_{\mathbb{N}}\int_{\mathbb{N}} |x_k|\,\mathbb{1}_{A_j}(k)\,\mu(dk)\,\mu(dj)\\
&= \int_{\mathbb{N}}\int_{\mathbb{N}} |x_k|\,\mathbb{1}_{A_j}(k)\,\mu(dj)\,\mu(dk)\\
&= \int_{\mathbb{N}} |x_k| \underbrace{\left(\int_{\mathbb{N}} \mathbb{1}_{A_j}(k)\,\mu(dj)\right)}_{=1,\ \text{as the } A_j \text{ are disjoint}} \mu(dk)\\
&= \int_{\mathbb{N}} |x_k|\,\mu(dk)\\
&= \sum_{k\in\mathbb{N}} |x_k|.
\end{aligned}$$

Problem 13.9 (i) Set $U(a,b) := a-b$. Then

$$U(u(x),y)\mathbb{1}_{[0,\infty)}(y) \geqslant 0 \iff u(x) \geqslant y \geqslant 0$$

and $U(u(x),y)\mathbb{1}_{[0,\infty)}(y)$ is a combination/sum/product of $\mathcal{B}(\mathbb{R}^2)$ resp. $\mathcal{B}(\mathbb{R})$-measurable functions. Thus $S[u]$ is $\mathcal{B}(\mathbb{R}^2)$-measurable.

(ii) Yes, true, since by Tonelli's theorem

$$\begin{aligned}\lambda^2(S[u]) &= \int_{\mathbb{R}^2} \mathbb{1}_{S[u]}(x,y)\,\lambda^2(d(x,y))\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{(x,y)\,:\,u(x)\geqslant y\geqslant 0\}}(x,y)\,\lambda^1(dy)\,\lambda^1(dx)\\ &= \int_{\mathbb{R}}\int_{[0,u(x)]} 1\,\lambda^1(dy)\,\lambda^1(dx)\\ &= \int_{\mathbb{R}} u(x)\,\lambda^1(dx)\end{aligned}$$

(iii) Measurability follows from (i) and with the hint. Moreover,

$$\begin{aligned}\lambda^2(\Gamma[u]) &= \int_{\mathbb{R}^2} \mathbb{1}_{\Gamma[u]}(x,y)\,\lambda^2(d(x,y))\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{(x,y)\,:\,y=u(x)\}}(x,y)\,\lambda^1(dy)\,\lambda^1(dx)\\ &= \int_{\mathbb{R}}\int_{[u(x),u(x)]} 1\,\lambda^1(dy)\,\lambda^1(dx)\\ &= \int_{\mathbb{R}} \lambda^1(\{u(x)\})\,\lambda^1(dx)\\ &= \int_{\mathbb{R}} 0\,\lambda^1(dx)\\ &= 0.\end{aligned}$$

Problem 13.10 The hint given in the text should be good enough to solve this problem....

Problem 13.11 Since (i) implies (ii), we will only prove (i) under the assumption that both $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are complete measure spaces. Note that we have to assume σ-finiteness of μ and ν, otherwise the product construction would not work. Pick some set $Z \in \mathcal{P}(X)\setminus\mathcal{A}$ (which is, because of completeness, not a null-set!), and some ν-null set $N \in \mathcal{B}$ and consider $Z\times N$.

We get for some exhausting sequence $(A_k)_k \subset \mathcal{A}$, $A_k \uparrow X$ and $\mu(A_k) < \infty$:

$$\begin{aligned}\mu\times\nu(X\times N) &= \sup_{k\in\mathbb{N}} \mu\times\nu(A_k\times N)\\ &= \sup_{k\in\mathbb{N}} \Big(\underbrace{\mu(A_k)}_{<\infty}\cdot\underbrace{\nu(N)}_{=0}\Big)\\ &= 0;\end{aligned}$$

thus $Z\times N \subset X\times N$ is a subset of a measurable $\mu\times\nu$ null set, hence it should be $\mathcal{A}\otimes\mathcal{B}$-measurable, if the product space were complete. On the other hand, because of Theorem 13.10(iii), if $Z\times N$ is $\mathcal{A}\otimes\mathcal{B}$-measurable, then the section

$$x \mapsto \mathbb{1}_{Z\times N}(x,y) = \mathbb{1}_Z(x)\mathbb{1}_N(y) \stackrel{y\in N}{=} \mathbb{1}_Z(x)$$

is $\mathcal{A}$-measurable which is only possible if $Z \in \mathcal{A}$.

Problem 13.12 (i) Let $A \in \mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$, fix $k \in \mathbb{N}$ and consider

$$\mathbb{1}_A(x,k) \quad \text{and} \quad B_k := \{x : \mathbb{1}_A(x,k) = 1\};$$

because of Theorem 13.10(iii), $B_k \in \mathcal{B}[0,\infty)$. Since

$$\begin{aligned}(x,k) \in A &\iff \mathbb{1}_A(x,k) = 1\\ &\iff \exists\, k \in \mathbb{N} : \mathbb{1}_A(x,k) = 1\\ &\iff \exists\, k \in \mathbb{N} : x \in B_k\end{aligned}$$

it is clear that $A = \bigcup_{k\in\mathbb{N}} B_k \times \{k\}$.

(ii) Let $M \in \mathcal{P}(\mathbb{N})$ and set $\zeta := \sum_{j\in\mathbb{N}} \delta_j$; we know that ζ is a (σ-finite) measure on $\mathcal{P}(\mathbb{N})$. Using Tonelli's theorem 13.8 we get

$$\begin{aligned}\pi(B \times M) &:= \sum_{m\in M} \pi(B \times \{m\})\\ &:= \sum_{m\in M} \int_B e^{-t}\frac{t^m}{m!}\,\mu(dt)\\ &= \int_M \int_B e^{-t}\frac{t^m}{m!}\,\mu(dt)\,\zeta(dm)\\ &= \iint_{B\times M} e^{-t}\frac{t^m}{m!}\,\mu\times\zeta(dt,dm)\end{aligned}$$

which shows that the measure $\pi(dt,dm) := e^{-t}\frac{t^m}{m!}\,\mu\times\zeta(dt,dm)$ has all the properties required by the exercise.

The uniqueness follows, however, from the uniqueness theorem for measures (Theorem 5.7): the family of 'rectangles' of the form $B \times M \in \mathcal{B}[0,\infty) \times \mathcal{P}(\mathbb{N})$ is a $\cap$-stable generator of the product σ-algebra $\mathcal{B}[0,\infty) \otimes \mathcal{P}(\mathbb{N})$ and contains an exhausting sequence, say, $[0,\infty) \times \{1,2,\dots k\} \uparrow [0,\infty) \times \mathbb{N}$. But on this generator π is (uniquely) determined by prescribing the values $\pi(B \times \{m\})$.

Problem 13.13 (i) This is similar to Problem 7.9, in particular (i) and (vi).

(ii) Note that

$$\begin{aligned}\mathbb{1}_B(x,y) &= \mathbb{1}_{(a,b]}(x)\mathbb{1}_{[x,b]}(y)\\ &= \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(a,y]}(x)\\ &= \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(a,b]}(y)\mathbb{1}_{[0,\infty)}(y-x);\end{aligned}$$

the last expression is, however, a product of (combinations of) measurable functions, thus $\mathbb{1}_B$ is measurable and so is then B.

Without loss of generality we can assume that $a > 0$, all other cases are similar.

Using Tonelli's theorem 13.8 we get

$$\mu \times \nu(B) = \iint \mathbb{1}_B(x,y)\,\mu\times\nu(dx,dy)$$

$$
\begin{aligned}
&= \iint \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(a,y]}(x)\,\mu\times\nu(dx,dy)\\
&= \int_{(a,b]}\int_{(a,y]}\mu(dx)\,\nu(dy)\\
&= \int_{(a,b]}\mu(a,y]\,\nu(dy)\\
&= \int_{(a,b]}\big(\mu(0,y]-\mu(0,a]\big)\,\nu(dy)\\
&= \int_{(a,b]}\mu(0,y]\nu(dy)-\mu(0,a]\int_{(a,b]}\nu(dy)\\
&= \int_{(a,b]}F(y)\,dG(y)-F(a)\big(G(b)-G(a)\big). \qquad (*)
\end{aligned}
$$

We remark at this point already that a very similar calculation (with μ,ν and F,G interchanged and with an open interval rather than a semi-open interval) yields

$$
\begin{aligned}
&\iint \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(y,b]}(x)\,\nu(dy)\,\mu(dx)\\
&\qquad= \int_{(a,b]}G(y-)\,dF(y)-G(a)\big(F(b)-F(a)\big). \qquad (**)
\end{aligned}
$$

(iii) On the one hand we have

$$
\begin{aligned}
\mu\times\nu\big((a,b]\times(a,b]\big) &= \mu(a,b]\nu(a,b]\\
&= \big(F(b)-F(a)\big)\big(G(b)-G(a)\big) \qquad (+)
\end{aligned}
$$

and on the other we find, using Tonelli's theorem at step (T)

$$
\begin{aligned}
&\mu\times\nu\big((a,b]\times(a,b]\big)\\
&= \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(a,b]}(y)\,\mu(dx)\,\nu(dy)\\
&= \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(x,b]}(y)\,\mu(dx)\,\nu(dy)+\\
&\qquad + \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(a,x)}(y)\,\mu(dx)\,\nu(dy)\\
&\overset{T}{=} \iint \mathbb{1}_{(a,b]}(x)\mathbb{1}_{(x,b]}(y)\,\mu(dx)\,\nu(dy)+\\
&\qquad + \iint \mathbb{1}_{(a,b]}(y)\mathbb{1}_{(y,b]}(x)\,\nu(dy)\,\mu(dx)\\
&\overset{*,**}{=} \int_{(a,b]}F(y)\,dG(y)-F(a)\big(G(b)-G(a)\big)+\\
&\qquad + \int_{(a,b]}G(y-)\,dF(y)-G(a)\big(F(b)-F(a)\big).
\end{aligned}
$$

Combining this formula with the previous one marked (+) reveals that

$$
F(b)G(b)-F(a)G(a) = \int_{(a,b]}F(y)\,dG(y)+\int_{(a,b]}G(y-)\,dF(y).
$$

Finally, observe that

$$
\begin{aligned}
\int_{(a,b]}\big(F(y)-F(y-)\big)\,dG(y) &= \int_{(a,b]}\mu(\{y\})\,\nu(dy)\\
&= \sum_{a<y\leqslant b}\mu(\{y\})\nu(\{y\})
\end{aligned}
$$

$$= \sum_{a<y\leqslant b} \Delta F(y)\Delta G(y).$$

(Mind that the sum is at most countable because of Lemma 13.12) from which the claim follows.

(iv) It is clear that uniform approximation allows to interchange limiting and integration procedures so that we *really* do not have to care about this. We show the formula for monomials $t, t^2, t^3, \ldots$ by induction. Write $\phi_n(t) = t^n$, $n \in \mathbb{N}$.

Induction start $n = 1$: in this case $\phi_1(t) = t$, $\phi_1'(t) = 1$ and $\phi(F(s)) - \phi(F(s-)) - \Delta F(s) = 0$, i.e. the formula just becomes

$$F(b) - F(a) = \int_{(a,b]} dF(s)$$

which is obviously true.

Induction assumption: for some n we know that

$$\phi_n(F(b)) - \phi_n(F(a)) = \int_{(a,b]} \phi_n'(F(s-))\,dF(s) + \sum_{a<s\leqslant b}\Big[\phi_n(F(s)) - \phi_n(F(s-)) - \phi_n'(F(s-))\Delta F(s)\Big].$$

Mind the misprint in the statement of the problem!

Induction step $n \rightsquigarrow n+1$: Write, for brevity $F = F(s)$ and $F_- = F(s-)$. We have because of (iii) with $G = \phi_n \circ F$ and because of the induction assumption

$$\begin{aligned}
&\phi_{n+1}(F(b)) - \phi_{n+1}(F(a)) \\
&= F(b)\phi_n(F(b)) - F(a)\phi_n(F(a)) \\
&= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,dF^n + \sum \Delta F\Delta F^n \\
&= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,\phi_n'(F_-)\,dF + \\
&\qquad + \sum\Big[F_-\phi_n(F) - F_-\phi_n(F_-) - F_-\phi_n'(F_-)\Delta F\Big] + \sum \Delta F\Delta F^n \\
&= \int_{(a,b]} F_-^n\,dF + \int_{(a,b]} F_-\,nF_-^{n-1}\,dF + \\
&\qquad + \sum\Big[F_-F^n - F_-^{n+1} - F_-nF_-^{n-1}\Delta F + \Delta F\Delta F^n\Big] \\
&= \int_{(a,b]} (n+1)F_-^n\,dF + \sum\Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\Big] \\
&= \int_{(a,b]} \phi_{n+1}' \circ F_-\,dF + \sum\Big[F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n\Big]
\end{aligned}$$

The expression under the sum can be written as

$$\begin{aligned}
&F_-F^n - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n \\
&\quad = (F_- - F)F^n + F^{n+1} - F_-^{n+1} - nF_-^n\Delta F + \Delta F\Delta F^n
\end{aligned}$$

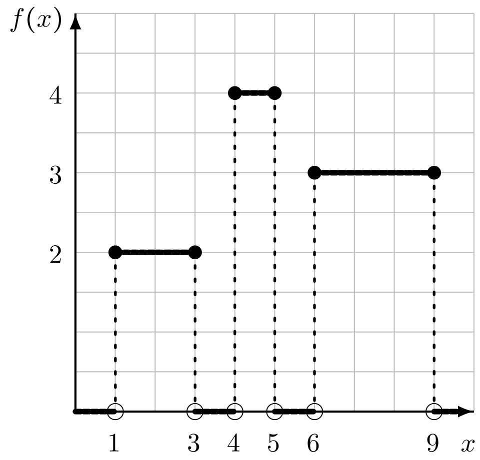
$$
\begin{aligned}
&= F^{n+1} - F_-^{n+1} + \Delta F\Big(- F^n - nF_-^n + \Delta F^n\Big) \\
&= F^{n+1} - F_-^{n+1} + \Delta F\Big(- F^n - nF_-^n + F^n - F_-^n\Big) \\
&= F^{n+1} - F_-^{n+1} - (n+1)F_-^n \Delta F \\
&= \phi_{n+1} \circ F - \phi_{n+1} \circ F_- - \phi'_{n+1} \circ F_- \Delta F
\end{aligned}
$$

and the induction is complete.

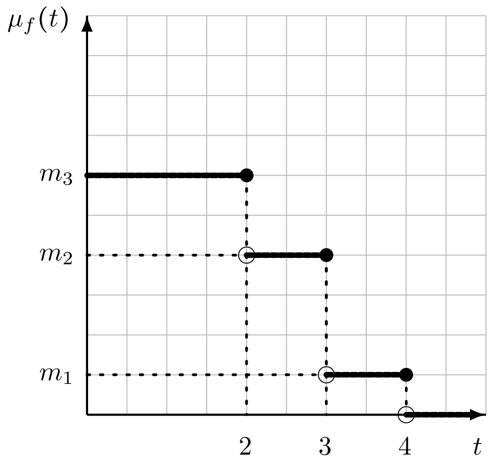
Problem 13.14 Mind the misprint in the problem:

$$\mu_f(t) := \mu(\{|f| \geqslant t\}).$$

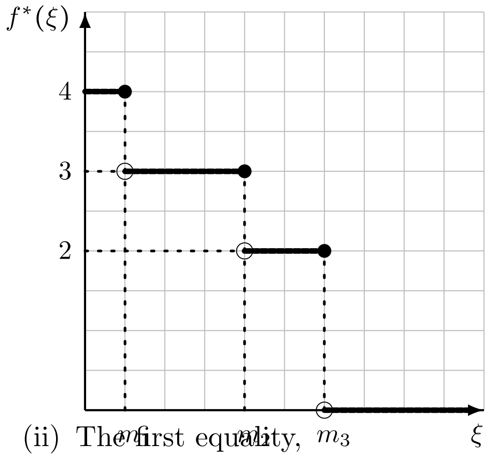
(i) We find the following pictures:



This is the graph of the original function $f(x)$.
Open and full dots indicate the continuity behaviour at the jump points.
x-values are to be measured in μ-length, i.e. x is a point in the measure space $(X, \mathcal{A}, \mu)$.



This is the graph of the associated distribution function $\mu_f(t)$. It is decreasing and left-continuous at the jump points.
t-values are to be measured using Lebesgue measure in $[0, \infty)$.
$m_1 = \mu\big([4,5]\big)$
$m_2 - m_1 = \mu\big([6,9]\big)$
$m_3 - m_2 = \mu\big([4,5]\big)$



This is the graph of the decreasing rearrangement $f^*(\xi)$ of $f(x)$. It is decreasing and left-continuous at the jump points. ξ-values are to be measured using Lebesgue measure in $[0,\infty)$. m_1, m_2, m_3 are as in the previous picture.

(ii) The first equality,

$$\int_{\mathbb{R}} |f|^p \, d\mu = p \int_0^\infty t^{p-1} \, \mu_f(t) \, dt,$$

follows immediately from Theorem 13.11 with $u = |f|$ and $\mu_f(t) = \mu(\{|f| \geqslant t\})$.

To show the second equality we have two possibilities. We can...

a) ...show the second equality first for (positive) elementary functions and use then a (by now standard...) Beppo Levi/monotone convergence argument to extend the result to all positive measurable functions. Assume that $f(x) = \sum_{j=0}^N a_j \mathbb{1}_{B_j}(x)$ is a positive elementary function in standard representation, i.e. $a_0 = 0 < a_1 < \cdots < a_n < \infty$ and the sets $B_j = \{f = a_j\}$ are pairwise disjoint. Then we have

$$\begin{aligned}
\mu(\{f = a_j\}) &= \mu(\{f \geqslant a_j\} \setminus \{f \geqslant a_{j+1}\}) \\
&= \mu(\{f \geqslant a_j\}) - \mu(\{f \geqslant a_{j+1}\}) \\
&= \mu_f(a_j) - \mu_f(a_{j+1}) \qquad\qquad (a_{n+1} := \infty, \mu_f(a_{n+1}) = 0) \\
&= \lambda^1\big((\mu_f(a_{j+1}), \mu_f(a_j)]\big) \\
&= \lambda^1(f^* = a_j).
\end{aligned}$$

This proves

$$\int f^p \, d\mu = \sum_{j=0}^n a_j^p \, \mu(B_j) = \sum_{j=0}^n a_j^p \, \lambda^1(f^* = a_j) = \int (f^*)^p \, d\lambda^1$$

and the general case follows from the above-mentioned Beppo Levi argument.

or we can

b) use Theorem 13.11 once again with $u = f^*$ and $\mu = \lambda^1$ provided we know that

$$\mu\big(\{|f| \geqslant t\}\big) = \lambda^1\big(\{f^* \geqslant t\}\big).$$

This, however, follows from

$$\begin{aligned}
f^*(\xi) \geqslant t &\iff \inf\{s : \mu_f(s) \leqslant \xi\} \geqslant t \\
&\iff \mu_f(t) \leqslant \xi \qquad\qquad (\text{as } \mu_f \text{ is left cts. \& decreasing})
\end{aligned}$$

$$\iff \mu\big(\{|f| \geqslant t\}\big) \leqslant \xi$$

and therefore

$$\lambda^1\big(\{\xi \,:\, f^*(\xi) \geqslant t\}\big) = \lambda^1\big(\{\xi \,:\, \mu(|f| \geqslant t) \leqslant \xi\}\big) = \mu(|f| \geqslant t).$$

14 Integrals with respect to image measures

Solutions to Problems 14.1–14.11

Problem 14.1 The first equality

$$\int u \, d(T(f\,\mu)) = \int u \circ T \, f \, d\mu$$

is just Theorem 14.1 combined with Lemma 10.8 the formula for measures with a density.

The second equality

$$\int u \circ T \, f \, d\mu = \int u \, f \circ T^{-1} \, dT(\mu)$$

is again Theorem 14.1.

The third equality finally follows again from Lemma 10.8.

Problem 14.2 We have for any $C \in \mathcal{B}$

$$\begin{aligned}
T(\mu)|_B(C) &= T(\mu)(B \cap C) \\
&= \mu\big(T^{-1}(B \cap C)\big) \\
&= \mu\big(T^{-1}(B) \cap T^{-1}(C)\big) \\
&= \mu\big(A \cap T^{-1}(C)\big) \\
&= \mu|_A\big(T^{-1}(C)\big) \\
&= T(\mu|_A)(C).
\end{aligned}$$

Problem 14.3 By definition, we find for any Borel set $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}
\delta_x \star \delta_y(B) &= \iint \mathbb{1}_B(s+t) \, \delta_x(ds) \, \delta_y(dt) \\
&= \int \mathbb{1}_B(x+t) \, \delta_y(dt) \\
&= \mathbb{1}_B(x+y) \\
&= \int {}_B(z) \, \delta_{x+y}(dz)
\end{aligned}$$

which means that $\delta_x \star \delta_y = \delta_{x+y}$. Note that, by Tonelli's theorem the order of the iterated integrals is irrelevant.

Similarly, since $z + t \in B \iff t \in B - z$, we find

$$\delta_z \star \mu(B) = \iint \mathbb{1}_B(s+t) \, \delta_z(ds) \, \mu(dt)$$

$$
\begin{aligned}
&= \int \mathbb{1}_B(z+t)\,\mu(dt) \\
&= \int \mathbb{1}_{B-z}(t)\,\mu(dt) \\
&= \mu(B-z) \\
&= \tau_{-z}(\mu)(B)
\end{aligned}
$$

where $\tau_z(t) := \tau(t-z)$ is the shift operator so that $\tau_{-z}^{-1}(B) = B-z$.

Problem 14.4 Since $x+y \in B \iff x \in B-y$, we can rewrite formula (14.9) in the following way:

$$
\begin{aligned}
\mu \star \nu(B) &= \iint \mathbb{1}_B(x+y)\,\mu(dx)\,\nu(dy) \\
&= \int \left[\int \mathbb{1}_{B-y}(x)\,\mu(dx) \right] \nu(dy) \\
&= \int \mu(B-y)\,\nu(dy).
\end{aligned}
$$

Similarly we get

$$
\mu \star \nu(B) = \int \mu(B-y)\,\nu(dy) = \int \nu(B-x)\,\mu(dx).
$$

Thus, if μ has no atoms, i.e. if $\mu(\{z\}) = 0$ for all $z \in \mathbb{R}^n$, we find

$$
\mu \star \nu(\{z\}) = \int \mu\big(\{z\}-y\big)\,\nu(dy) = \int \mu\big(\underbrace{\{z-y\}}_{=0}\big)\,\nu(dy) = 0.
$$

Problem 14.5 Because of Tonelli's theorem we can iterate the very definition of 'convolution' of two measures, Definition 14.4(iii), and get

$$
\mu_1 \star \cdots \star \mu_n(B) = \int \cdots \int \mathbb{1}_B(x_1 + \cdots + x_n)\,\mu_1(dx_1) \cdots \mu_n(dx_n)
$$

so that the formula derived at the end of Remark 14.5(ii), page 138, applies and yields

$$
\begin{aligned}
&\int |\omega|\,P^{\star n}(d\omega) \\
&\quad= \int \cdots \int |\omega_1 + \omega_2 + \cdots + \omega_n|\,P(d\omega_1)\,P(d\omega_2) \cdots P(d\omega_n) \\
&\quad\overset{*}{\leqslant} \int \cdots \int \Big(|\omega_1| + |\omega_2| + \cdots + |\omega_n|\Big)\,P(d\omega_1)\,P(d\omega_2) \cdots P(d\omega_n) \\
&\quad= \sum_{j=1}^n \int \cdots \int |\omega_j|\,P(d\omega_1)\,P(d\omega_2) \cdots P(d\omega_n) \\
&\quad= \sum_{j=1}^n \int |\omega_j|\,P(d\omega_j) \cdot \prod_{k \neq j} \int P(d\omega_k) \\
&\quad= \sum_{j=1}^n \int |\omega_j|\,P(d\omega_j) \\
&\quad= n \int |\omega_1|\,P(d\omega_1)
\end{aligned}
$$

where we used the symmetry of the iterated integrals in the integrating measures as well as the fact that $P(\mathbb{R}^n) = \int P(d\omega_k) = 1$. Note that we can have $+\infty$ on either side.

The equality $\int \omega\, P^{\star n}(d\omega) = n\int \omega\, P(d\omega)$ follows with same calculation (note that we do not get an inequality as there is no need for the triangle inequality at point (*) above). The integrability condition is now needed since the integrands are no longer positive. Note that, since $\omega \in \mathbb{R}^n$, the above equality is an equality between vectors in $\mathbb{R}^n$; this is no problem, just read the equality coordinate-by-coordinate.

Problem 14.6 Since the convolution $p \mapsto u \star p$ is linear, it is enough to consider monomials of the form $p(x) = x^k$. Thus, by the binomial formula,

$$\begin{aligned} u \star p(x) &= \int u(x-y)\, y^k\, dy \\ &= \int u(y)\, (x-y)^k\, dy \\ &= \sum_{j=0}^{k} \binom{k}{j} x^j \int u(y)\, y^{k-j}\, dy. \end{aligned}$$

Since $\operatorname{supp} u$ is compact, there is some $r > 0$ such that $\operatorname{supp} u \subset B_r(0)$ and we get for any $m \in \mathbb{N}_0$, and in particular for $m = k - j$ or $m = k$, that

$$\begin{aligned} \left| \int u(y) y^m\, dy \right| &\leqslant \int_{\operatorname{supp} u} \|u\|_\infty |y|^m\, dy \\ &\leqslant \int_{B_r(0)} \|u\|_\infty r^m\, dy \\ &= 2r \cdot r^m \cdot \|u\|_\infty \end{aligned}$$

which is clearly finite. This shows that $u \star p$ exists and that it is a polynomial.

Problem 14.7 That the convolution $u \star w$ is bounded and continuous follows from Theorem 14.8.

Monotonicity follows from the monotonicity of the integral: if $x \leqslant z$, then

$$u \star w(x) = \int \underbrace{u(y)}_{\geqslant 0} \cdot \underbrace{w(x-y)}_{\leqslant w(z-y)}\, dy \leqslant \int u(y) \cdot w(z-y)\, dy = u \star w(y).$$

Problem 14.8 (This solution is written for $u \in C_c(\mathbb{R}^n)$ and $w \in C^\infty(\mathbb{R}^n)$).

Let $\partial_j = \partial/\partial x_j$ denote the partial derivative in direction x_j where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Since

$$w \in C^\infty \implies \partial_j w \in C^\infty,$$

it is enough to show $\partial_j(u \star w) = u \star \partial_j w$ and to iterate this equality. In particular, we find $\partial^\alpha(u \star w) = u \star \partial^\alpha w$ where

$$\partial^\alpha = \frac{\partial^{\alpha_1 + \cdots \alpha_n}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}, \quad \alpha \in \mathbb{N}_0^n.$$

Since u has compact support and since the derivative is a local operation (i.e., we need to know a function only in a neighbourhood of the point where we differentiate), and since we have for any $r > 0$

$$\sup_{y \in \operatorname{supp} u} \sup_{x \in B_r(0)} \left| \frac{\partial}{\partial x_j} w(x-y) \right| \leqslant c(r),$$

we can use the differentiability lemma for parameter-dependent integrals, Theorem 11.5 to find for any $x \in B_{r/2}(0)$, say,

$$\begin{aligned}\frac{\partial}{\partial x_j} \int u(y) w(x-y)\, dy &= \int u(y) \frac{\partial}{\partial x_j} w(x-y)\, dy \\ &= \int u(y) \big(\tfrac{\partial}{\partial x_j} w\big)(x-y)\, dy \\ &= u \star \partial_j w(x).\end{aligned}$$

Problem 14.9 The measurability considerations are just the same as in Theorem 14.6, so we skip this part.

By assumption,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r};$$

We can rewrite this as

$$\frac{1}{r} + \underbrace{\left[\frac{1}{p} - \frac{1}{r}\right]}_{=1-\frac{1}{q}\in[0,1)} + \underbrace{\left[\frac{1}{q} - \frac{1}{r}\right]}_{=1-\frac{1}{p}\in[0,1)} = 1. \tag{*}$$

Now write the integrand appearing in the definition of $u \star w(x)$ in the form

$$|u(x-y)w(y)| = \Big[|u(x-y)|^{p/r}|w(y)|^{q/r}\Big] \cdot \Big[|u(x-y)|^{1-p/r}\Big] \cdot \Big[|w(y)|^{1-q/r}\Big]$$

and apply the generalized Hölder inequality (cf. Problem 12.4) with the exponents from (*):

$$\begin{aligned}|u \star w(x)| &\leqslant \int |u(x-y)w(y)|\, dy \\ &\leqslant \left[\int |u(x-y)|^p |w(y)|^q\, dy\right]^{\frac{1}{r}} \left[\int |u(x-y)|^p\, dy\right]^{\frac{1}{p}-\frac{1}{r}} \left[\int |w(y)|^p\, dy\right]^{\frac{1}{q}-\frac{1}{r}}.\end{aligned}$$

Raising this inequality to the rth power we get, because of the translation invariance of Lebesgue measure,

$$\begin{aligned}|u \star w(x)|^r &\leqslant \left[\int |u(x-y)|^p |w(y)|^q\, dy\right] \|u\|_p^{r-p} \cdot \|w\|_q^{r-q} \\ &= |u|^p \star |w|^q(x) \cdot \|u\|_p^{r-p} \cdot \|w\|_q^{r-q}.\end{aligned}$$

Now we integrate this inequality over x and use Theorem 14.6 for $p = 1$ and the integral

$$\int |u|^p \star |w|^q(x)\, dx = \||u|^p \star |w|^q\|_1 \leqslant \|u\|_p^p \cdot \|w\|_q^q.$$

Thus,

$$\|u \star w\|_r^r = \int |u \star w(x)|^r\, dx \leqslant \|u\|_p^p \cdot \|w\|_q^q \cdot \|u\|_p^{r-p} \cdot \|w\|_q^{r-q} = \|u\|_p^r \cdot \|w\|_q^r$$

and the claim follows.

Problem 14.10 (i) Since ϕ is rotationally invariant, it is enough to show that the function

$$\psi(r) := e^{1/(r^2-1)}\mathbb{1}_{[-1,1]}(r)$$

is of class C^∞. This is a standard argument and we only sketch it here. Clearly, the critical points are $r = \pm 1$. Since $\psi(\pm 1) = 0 = e^{-1/0} = e^{-\infty} = 0$, the function ψ is continuous. Differentiability is shown using induction:

$$\psi'(r) = \frac{2r}{1-r^2}\, e^{1/(r^2-1)}\mathbb{1}_{[-1,1]}(r)$$

and if $\psi^{(k)}(r) = f_k(r)e^{1/(r^2-1)}\mathbb{1}_{[-1,1]}(r)$, then

$$\psi^{(k+1)}(r) = f_k'(r)e^{1/(r^2-1)}\mathbb{1}_{[-1,1]}(r) + f_k(r)\frac{2r}{1-r^2}\, e^{1/(r^2-1)}\mathbb{1}_{[-1,1]}(r).$$

This shows that $f_k(r)$ is for all $k \in \mathbb{N}$ a rational function whose growth to $\pm\infty$ as $r \to \pm 1$ is not so strong as the decay of $e^{1/(r^2-1)}$ to 0 as $r \to \pm 1$. This proves that $\psi(r)$ is arbitrarily often differentiable at the points $r = \pm 1$ (with zero derivative). For all $r \neq \pm 1$ the situation is clear.

The constant κ^{-1} is necessarily the integral of the function ϕ:

$$\kappa^{-1} = \int_{B_1(0)} \exp\left[\frac{1}{|x|^2 - 1}\right] dx\,.$$

(ii) That ϕ_ϵ is a C^∞-function is clear, since ϕ_ϵ is constructed from ϕ by a dilation.

Clearly,

$$\phi_\epsilon(x) = 0 \iff \phi(x/\epsilon) = 0 \iff |x/\epsilon| \geqslant 1 \iff |x| \geqslant \epsilon.$$

This means that $\operatorname{supp}\phi_\epsilon = \overline{B_\epsilon(0)}$.

Using Theorem 14.1 for the dilation $T = T_{1/\epsilon} : x \mapsto x/\epsilon$ and, cf. Problem 5.8 or Theorem 7.10, the fact that for Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned} T_{1/\epsilon}(\lambda^n)(B) &\stackrel{\text{def}}{=} \lambda^n(T_{1/\epsilon}^{-1}(B)) \\ &= \lambda^n(T_\epsilon(B)) \\ &= \lambda^n(\epsilon \cdot B) \\ &\stackrel{7.10}{=} \epsilon^n \cdot \lambda^n(B), \end{aligned}$$

we get

$$\begin{aligned} \int \phi_\epsilon(x)\, \lambda^n(dx) &= \epsilon^{-n} \int \phi(T_{1/\epsilon}(x))\, \lambda^n(dx) \\ &= \epsilon^{-n} \int \phi(x)\, T_{1/\epsilon}(\lambda^n)(dx) \\ &= \epsilon^{-n} \int \phi(x)\, \epsilon^n \cdot \lambda^n(dx) \\ &= \int \phi(x)\, \lambda^n(dx). \end{aligned}$$

(iii) We show, more generally, that

$$\operatorname{supp} u \star w \subset \operatorname{supp} u + \operatorname{supp} w \tag{*}$$

whenever $u \star w$ makes sense. Now

$$\int u(x-y)w(y)\,dy = \int_{\operatorname{supp} w} u(x-y)w(y)\,dy$$

so that

$$x - y \notin \operatorname{supp} u \iff x \notin y + \operatorname{supp} u\,.$$

Thus,

$$x \notin \operatorname{supp} u + \operatorname{supp} w \implies u \star w(x) = 0.$$

Since $\operatorname{supp} u + \operatorname{supp} w$ is a closed set, we have shown (*).

(iv) The estimate

$$\|\phi_\epsilon \star u\|_p \leqslant \|\phi_\epsilon\|_1 \cdot \|u\|_p \tag{**}$$

follows from Theorem 14.6.

Since $\phi_\epsilon \in C_c^\infty \implies \partial^\alpha \phi_\epsilon \in C_c^\infty$ for any $\alpha \in \mathbb{N}_0^n$. This means that $u \star \partial^\alpha \phi_\epsilon$ is well defined. However, if $p \neq 1$, we cannot appeal naively to the differentiability lemma, Theorem 11.5 to swap integration (i.e. convolution) and differentiation. To do this we consider the sequence

$$u_k(x) := \big((-k) \vee u(x) \wedge k\big)\mathbb{1}_{B_k(0)}(x)$$

and note that, by dominated convergence, $L^p\text{-}\lim_k u_k = u$ while $u_k \in L^1 \cap L^\infty$. In this setting we can apply Theorem 11.5 and get

$$\begin{aligned}
\partial^\alpha\big(\phi_\epsilon \star u_k(x)\big) &= \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}\int \phi_\epsilon(x-y)u_k(y)\,dy\\
&= \int \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}\phi_\epsilon(x-y)u_k(y)\,dy\\
&= \big(\partial^\alpha\phi_\epsilon\big) \star u_k(x).
\end{aligned}$$

(Note that ϕ_ϵ and u_k have compact support and are bounded functions, so domination is no problem at all.) Using the estimate (**) we find that

$$\begin{aligned}
\|\partial^\alpha\big(\phi_\epsilon \star (u_k - u_\ell)\big)\|_p &= \|\big(\partial^\alpha\phi_\epsilon\big) \star (u_k - u_\ell)\|_p\\
&\leqslant \|\partial^\alpha\phi_\epsilon\|_1 \cdot \|u_k - u_\ell\|_p \xrightarrow{k,\ell\to\infty} 0.
\end{aligned}$$

Since, similarly,

$$\|\big(\partial^\alpha\phi_\epsilon\big) \star (u_k - u)\|_p \leqslant \|\partial^\alpha\phi_\epsilon\|_1 \cdot \|u_k - u\|_p \xrightarrow{k\to\infty} 0,$$

we conclude that

$$\big(\partial^\alpha\phi_\epsilon\big) \star u_k \xrightarrow[L^p]{k\to\infty} \big(\partial^\alpha\phi_\epsilon\big) \star u$$

and

$$\partial^\alpha\big(\phi_\epsilon \star u_k\big) \xrightarrow[L^p]{k\to\infty} \partial^\alpha\big(\phi_\epsilon \star u\big)$$

so that

$$\partial^\alpha\big(\phi_\epsilon \star u\big) = \big(\partial^\alpha \phi_\epsilon\big) \star u.$$

(v) Since $\int \phi_\epsilon(y)\,dy = 1$, we get from Minkowski's inequality for integrals, Theorem 13.14,

$$\begin{aligned}
&\|u - u \star \phi_\epsilon\|_p \\
&= \left(\int \left| \int \big(u(x) - u(x-y)\big)\phi_\epsilon(y)\,dy \right|^p dx \right)^{1/p} \\
&\leqslant \int \|u(\cdot) - u(\cdot - y)\|_p\, \phi_\epsilon(y)\,dy \\
&= \left\{ \int\limits_{|y|\leqslant h} + \int\limits_{|y|>h} \right\} \|u(\cdot) - u(\cdot - y)\|_p\, \phi_\epsilon(y)\,dy.
\end{aligned}$$

Since the integrand $y \mapsto \|u(\cdot) - u(\cdot - y)\|_p$ is continuous, cf. Theorem 14.8, we can, for a given $\delta > 0$, pick $h = h(\delta)$ in such a way that

$$\|u(\cdot) - u(\cdot - y)\|_p \leqslant \delta \qquad \forall\, |y| \leqslant h.$$

Thus, using this estimate for the first integral term, and the triangle inequality in L^p and the translation invariance of Lebesgue integrals for the second integral expression, we get

$$\begin{aligned}
\|u - u \star \phi_\epsilon\|_p &\leqslant \int\limits_{|y|\leqslant h} \delta\, \phi_\epsilon(y)\,dy + \int\limits_{|y|>h} 2\|u\|_p\, \phi_\epsilon(y)\,dy \\
&\leqslant \delta \int \phi_\epsilon(y)\,dy + 2\|u\|_p \int\limits_{|y|>h} \phi_\epsilon(y)\,dy \\
&\leqslant \delta + 2\|u\|_p \int\limits_{|y|>h} \phi_\epsilon(y)\,dy.
\end{aligned}$$

Since $\operatorname{supp}\phi_\epsilon = \overline{B_\epsilon(0)}$, we can let $\epsilon \to 0$, and then $\delta \to 0$, and get

$$\limsup_{\epsilon\to 0} \|u - u \star \phi_\epsilon\|_p \leqslant \delta \xrightarrow{\delta\to 0} 0,$$

and the claim follows.

Problem 14.11 Note that $v(x) = \frac{d}{dx}(1 - \cos x)\mathbb{1}_{[0,2\pi)}(x) = \mathbb{1}_{(0,2\pi)}(x)\sin x$. Thus,

(i)

$$u \star v(x) = \int_0^{2\pi} 1_{\mathbb{R}}(x-y)\sin y\,dy = \int_0^{2\pi} \sin y\,dy = 0 \quad \forall\, x.$$

(ii) Since all functions u, v, w, ϕ are continuous, we can use the usual rules for the (Riemann) integral and get, using integration by parts and the fundamental theorem of integral calculus,

$$v \star w(x) = \int \tfrac{d}{dx}\phi(x-y) \int_{-\infty}^{x} \phi(t)\,dt\,dx$$

$$\begin{aligned}
&= \int \left(-\tfrac{d}{dy}\phi(x-y)\right) \int_{-\infty}^{y} \phi(t)\,dt\,dx \\
&= \int \phi(x-y)\,\tfrac{d}{dy}\int_{-\infty}^{y} \phi(t)\,dt\,dx \\
&= \int \phi(x-y)\,\phi(y)\,dy \\
&= \phi \star \phi(x).
\end{aligned}$$

If $x \in (0,4\pi)$, then $x-y \in (0,2\pi)$ for some suitable $y = y_=$ and even for all y from an interval $(y_0-\epsilon, y_0+\epsilon) \subset (0,2\pi)$. Since ϕ is positive with support $[0,2\pi]$, the positivity follows.

(iii) Obviously,

$$(u \star v) \star w \overset{(i)}{=} 0 \star w = 0$$

while

$$\begin{aligned}
u \star (v \star w)(x) &= \int 1_{\mathbb{R}}(x-y) v \star w(y)\,dy \\
&= \int v \star w(y)\,dy \\
&= \int \phi \star \phi(y)\,dy \\
&> 0.
\end{aligned}$$

Note that w is not an (pth power, $p < \infty$) integrable function so that we cannot use Fubini's theorem to prove associativity of the convolution.

15 Integrals of images and Jacobi's transformation rule Solutions to Problems 15.1–15.10

Problem 15.1 Since F and F_j are F_σ-sets, we get

$$F = \bigcup_{k\in\mathbb{N}} C_k, \qquad F_j = \bigcup_{k\in\mathbb{N}} C_k^j$$

for closed sets C_k resp. C_k^j. Since complements of closed sets are open, we find, using the rules for (countable) unions and intersections that

(i) $$\bigcap_{j=1}^n F_j = \bigcap_{j=1}^n \bigcup_{k\in\mathbb{N}} C_k^j = \bigcup_{k\in\mathbb{N}} \underbrace{\bigcap_{j=1}^n C_k^j}_{\text{closed set}}.$$

(ii) $$\bigcup_{j\in\mathbb{N}} F_j = \bigcup_{j\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} C_k^j = \underbrace{\bigcup_{(j,k)\in\mathbb{N}\times\mathbb{N}} C_k^j}_{\text{countable union!}}.$$

Moreover, $$\bigcap_{j\in\mathbb{N}} F_j^c = \bigcap_{j\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \left[C_k^j\right]^c = \underbrace{\bigcap_{(j,k)\in\mathbb{N}\times\mathbb{N}} \left[C_k^j\right]^c}_{\text{countable intersection!}}.$$

(iii) $$F = \bigcup_{k\in\mathbb{N}} C_k \implies F^c = \left[\bigcup_{k\in\mathbb{N}} C_k\right]^c = \bigcap_{k\in\mathbb{N}} \underbrace{C_k^c}_{\text{open}}.$$

(iv) Set $c_1 := C$ and $C_j = \varnothing$, $j \geqslant 2$. Then $C = \bigcup_{j\in\mathbb{N}} C_j$ is an F_σ-set.

Problem 15.2 Write $\lambda = \lambda^n$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$. Fix $B \in \mathcal{B}$. According to Lemma 15.2 there are sets $F \in F_\sigma$ and $G \in G_\delta$ such that

$$F \subset B \subset G \quad\text{and}\quad \lambda(F) = \lambda(B) = \lambda(G).$$

Since for closed sets C_j and open sets U_j we have $F = \bigcup C_j$ and $G = \bigcap U_j$ we get for some $\epsilon > 0$ and suitable $M = M_\epsilon \in \mathbb{N}$, $N = N_\epsilon \in \mathbb{N}$ that

$$C_1 \cup \cdots \cup C_N \subset B \subset U_1 \cap \cdots \cap U_M$$

and

$$\left|\lambda(U_1 \cap \cdots \cap U_M) - \lambda(B)\right| \leqslant \epsilon, \tag{*}$$

$$\left|\lambda(B) - \lambda(C_1 \cup \cdots \cup C_N)\right| \leqslant \epsilon. \tag{**}$$

Since finite unions of closed sets are closed and finite intersections of open sets are open, (*) proves outer regularity while (**) proves inner regularity (w.r.t. close sets).

To see inner regularity with compact sets, we note that the closed set $C' := C_1 \cup \cdots \cup C_N$ is approximated by the following compact sets

$$K_\ell := \overline{B_\ell(0)} \cap C' \uparrow C' \quad \text{as} \quad \ell \to \infty$$

and, because of the continuity of measures, we get for suitably large $L = L_\epsilon \in \mathbb{N}$ that

$$\left|\lambda(K_L) - \lambda(C_1 \cup \cdots \cup C_N)\right| \leqslant \epsilon$$

which can be combined with (**) to give

$$\left|\lambda(K_L) - \lambda(B)\right| \leqslant 2\epsilon.$$

This shows inner regularity for the compact sets.

Problem 15.3 Notation (for brevity): Write $\lambda = \lambda^n$, $\bar\lambda = \overline{\lambda^n}$, $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}^* = \mathcal{B}^*(\mathbb{R}^n)$. By definition, $B^* = B \cup N^*$ where N^* is a subset of a $\mathcal{B}$-measurable null set N. (We indicate $\mathcal{B}^*$-sets by an asterisk, C (with and without ornaments and indices C' ...) is always a closed set and U etc. is always an open set.

Solution 1: Following the hint we get (with the notation of Problem 10.12)

$$\begin{aligned}
\lambda(B) = \bar\lambda(B^*) &= \lambda^*(B^*) \\
&= \inf_{\mathcal{B} \ni A \supset B^*} \lambda(A) && \text{(by 10.12)} \\
&= \inf_{\mathcal{B} \ni A \supset B^*} \inf_{U \supset A} \lambda(U) && \text{(by 15.2)} \\
&\leqslant \inf_{U' \supset B \cup N} \inf_{U \supset U'} \lambda(U) && \text{(as } B* \subset B \cup N) \\
&= \inf_{U' \supset B^*} \lambda(U') && \text{(by 15.2)} \\
&= \lambda(B \cup N) && \text{(by 15.2)} \\
&\leqslant \lambda(B) + \lambda(N) \\
&= \lambda(B).
\end{aligned}$$

Inner regularity (for closed sets) follows similarly,

$$\begin{aligned}
\lambda(B) = \bar\lambda(B^*) &= \lambda_*(B^*) \\
&= \sup_{\mathcal{B} \ni A \subset B^*} \lambda(A) && \text{(by 10.12)} \\
&= \sup_{\mathcal{B} \ni A \subset B^*} \sup_{C \subset A} \lambda(C) && \text{(by 15.2)} \\
&\geqslant \sup_{C' \subset B^*} \sup_{C \subset C'} \lambda(C) \\
&= \sup_{C' \subset B^*} \lambda(C') && \text{(by 15.2)}
\end{aligned}$$

$$\geqslant \sup_{C'\subset B} \lambda(C') \qquad \text{(as } B\subset B^*\text{)}$$

$$= \lambda(B), \qquad \text{(by 15.2)}$$

and inner regularity for compact sets is the same calculation.

There is a more elementary ....

Solution 2: (without Problem 10.12). Using the definition of the completion we get

$$\begin{aligned}\bar\lambda(B^*) = \lambda(B) &= \sup_{C'\subset B} \lambda(C')\\ &\leqslant \sup_{C\subset B^*} \lambda(C)\\ &\leqslant \sup_{C''\subset B\cup N} \lambda(C'')\\ &= \lambda(B\cup N)\\ &= \lambda(B)\end{aligned}$$

as well as

$$\begin{aligned}\bar\lambda(B^*) = \lambda(B) &= \inf_{U'\supset B} \lambda(U')\\ &\leqslant \inf_{U\supset B^*} \lambda(U)\\ &\leqslant \inf_{U''\supset B\cup N} \lambda(U'')\\ &= \lambda(B\cup N)\\ &= \lambda(B).\end{aligned}$$

Problem 15.4 (i) Obviously, $\mathcal{G}\subset\mathcal{B}[0,\infty)$. On the other hand, $\sigma(\mathcal{G})$ contains all open intervals of the form

$$(\alpha,\beta) = \bigcup_{n\in\mathbb{N}} \left[\alpha - \tfrac{1}{n}, \infty\right) \smallsetminus \left[\beta,\infty\right), \qquad 0\leqslant\alpha<\beta<\infty \tag{*}$$

and all intervals of the form

$$[0,\beta) = [0,\infty)\smallsetminus[\beta,\infty), \qquad \beta>0. \tag{**}$$

Thus,

$$\sigma(\mathcal{G}) \supset \mathcal{O}(\mathbb{R})\cap[0,\infty)$$

since any open set $U\in\mathcal{O}(\mathbb{R})$ is a countable union of open intervals,

$$U = \bigcup_{\substack{\alpha<\beta,\alpha,\beta\in\mathbb{Q}\\ (\alpha,\beta)\subset U}} (\alpha,\beta),$$

so that $U\cap[0,\infty)\in\mathcal{O}\cap[0,\infty)$ is indeed a countable union of sets of the form (*) and (**). Thus,

$$\mathcal{B}[0,\infty) = \sigma(\mathcal{O}\cap[0,\infty)) \subset \sigma(\mathcal{G}) \subset \mathcal{B}[0,\infty).$$

(ii) That μ is a measure follows from Lemma 10.8 (for a proof, see the online section 'additional material'). Since

$$\rho(B) = \mu(T_{1/5}^{-1}(B)) = T_{1/5}(\rho)(B)$$

where $T_{1/5}(x) = \frac{1}{5} \cdot x$, ρ is an image measure, hence a measure.

Since

$$\rho[a, \infty) = \mu[5a, \infty) \leqslant \mu[a, \infty) \qquad \forall\, a \geqslant 0,$$

we have $\rho\big|_{\mathcal{G}} \leqslant \mu\big|_{\mathcal{G}}$. On the other hand,

$$\rho\left[\tfrac{3}{5}, \tfrac{4}{5}\right) = \mu[3, 4) = 1 > 0 = \mu\left[\tfrac{3}{5}, \tfrac{4}{5}\right).$$

This does not contradict Lemma 15.6 since $\mathcal{G}$ is not a semi-ring.

Problem 15.5 We want to show that

a) $\lambda^n(x + B) = \lambda^n(B),\ B \in \mathcal{B}(\mathbb{R}^n),\ x \in \mathbb{R}^n$ (Theorem 5.8(i));

b) $\lambda^n(t \cdot B) = t^n \lambda^n(B),\ B \in \mathcal{B}(\mathbb{R}^n),\ t \geqslant 0$ (Problem 5.8);

c) $A(\lambda^n) = |\det A^{-1}| \cdot \lambda^n,\ A \in \mathrm{GL}(n, \mathbb{R})$ (Theorem 7.10).

From Theorem 15.5 we know that for any C^1 diffeomorphism ϕ the formula

$$\lambda^n(\phi(B)) = \int_B |\det D\phi|\, d\lambda^n$$

holds. Thus a), b), c) follow upon setting

a) $\phi(y) = x + y \implies D\phi \equiv 1 \implies |\det D\phi| \equiv 1;$

b) $\phi(y) = t \cdot y \implies D\phi \equiv t \cdot \mathrm{id} \implies |\det D\phi| \equiv t^n;$

c) $\phi(y) = A^{-1} y \implies D\phi(y) \equiv A^{-1} \implies |\det D\phi| \equiv |\det A|^{-1}.$

Problem 15.6 Mind the misprints in the statement of the problem.

(i) The map $\Phi : \mathbb{R} \ni x \mapsto (x, f(x))$ is obviously bijective and differentiable with derivative $D\Phi(x) = (x, f'(x))$ so that $|D\Phi(x)|^2 = 1 + (f'(x))^2$. The inverse of Φ is given by $\Phi^{-1} : (x, f(x)) \mapsto x$ which is clearly differentiable.

(ii) Since $|D\Phi(x)| = \sqrt{1 + (f'(x))^2}$ is positive and measurable, it is a density function and $\mu := |D\Phi(x)| \cdot \lambda$ is a measure, cf. Lemma 10.8, while $\sigma = \Phi(\mu)$ is an image measure in the sense of Definition 7.7.

(iii) This is Theorem 14.1 and/or Problem 14.1.

(iv) The normal is, by definition, orthogonal to the gradient: $D\Phi(x) = (1, f'(x))$; obviously $|n(x)| = 1$ and

$$n(x) \cdot D\Phi(x) = \frac{\begin{pmatrix} -f'(x) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}}{\sqrt{1 + (f'(x))^2}} = 0.$$

Further,

$$\tilde{\Phi}(x,r) = \begin{pmatrix} x - \frac{rf'(x)}{\sqrt{1+[f'(x)]^2}} \\ f(x) + \frac{r}{\sqrt{1+[f'(x)]^2}} \end{pmatrix},$$

so that

$$\begin{aligned} D\tilde{\Phi}(x,r) &= \left(\frac{\partial\tilde{\Phi}(x,r)}{\partial(x,r)}\right) \\ &= \begin{pmatrix} 1 - r\,\frac{\partial}{\partial x}\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} & f'(x) + r\,\frac{\partial}{\partial x}\frac{1}{\sqrt{1+[f'(x)]^2}} \\ -\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} & \frac{1}{\sqrt{1+[f'(x)]^2}} \end{pmatrix} \end{aligned}$$

For brevity we write f, f', f'' instead of $f(x), f'(x), f''(x)$. Now

$$\frac{\partial}{\partial x}\frac{f'(x)}{\sqrt{1+[f'(x)]^2}} = \frac{f''\sqrt{1+[f']^2} - f'\,\frac{f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}$$

and

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{1+[f'(x)]^2}} = \frac{-\frac{f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}.$$

Thus, $\det D\tilde{\Phi}(x,r)$ becomes

$$\begin{aligned} &\frac{1}{\sqrt{1+[f']^2}}\left(1 - \frac{r\,f''\sqrt{1+[f']^2} - \frac{r\,[f']^2 f''}{\sqrt{1+[f']^2}}}{1+[f']^2}\right) \\ &\qquad + \frac{f'}{\sqrt{1+[f']^2}}\left(f' - \frac{\frac{r\,f'f''}{\sqrt{1+[f']^2}}}{1+[f']^2}\right) \\ &= \frac{1}{\sqrt{1+[f']^2}} - \frac{r\,f'' - \frac{r\,[f']^2 f''}{1+[f']^2}}{1+[f']^2} + \frac{[f']^2}{\sqrt{1+[f']^2}} - \frac{\frac{r\,[f']^2 f''}{1+[f']^2}}{1+[f']^2} \\ &= \frac{1+[f']^2}{\sqrt{1+[f']^2}} - \frac{r\,f''}{1+[f']^2} \\ &= \sqrt{1+[f']^2} - \frac{r\,f''}{1+[f']^2} \end{aligned}$$

If x is from a compact set, say $[c,d]$, we can, because of the continuity of f, f' and f'', achieve that for sufficiently small values of $|r| < \epsilon$ we get that $\det D\tilde{\Phi} > 0$, i.e. $\tilde{\Phi}$ is a local C^1-diffeomorphism.

(v) The set is a 'tubular' neighbourhood of radius r around the graph Γ_f for $x \in [c,d]$. Measurability follows, since $\tilde{\Phi}$ is a diffeomorphism, from the fact that the set $C(r)$ is the image of the cartesian product of measurable sets.

(vi) Because of part (iv) we have, for fixed x and sufficiently small values of r, that the determinant is positive so that

$$\lim_{r\downarrow 0}\frac{1}{2r}\int_{(-r,r)}\left|\det D\tilde{\Phi}(x,s)\right|\lambda^1(ds)$$

$$
\begin{aligned}
&= \lim_{r\downarrow 0} \frac{1}{2r} \int_{(-r,r)} \left| \sqrt{1+(f'(x))^2} - \frac{s\,f''(x)}{1+(f'(x))^2} \right| \lambda^1(ds)\\
&= \lim_{r\downarrow 0} \frac{1}{2r} \int_{(-r,r)} \left(\sqrt{1+(f'(x))^2} - \frac{s\,f''(x)}{1+(f'(x))^2} \right) \lambda^1(ds)\\
&= \lim_{r\downarrow 0} \frac{1}{2r} \int_{(-r,r)} \sqrt{1+(f'(x))^2}\, \lambda^1(ds)\\
&\qquad - \lim_{r\downarrow 0} \frac{1}{2r} \int_{(-r,r)} \frac{s\,f''(x)}{1+(f'(x))^2}\, \lambda^1(ds)\\
&= \sqrt{1+(f'(x))^2} - \frac{f''(x)}{1+(f'(x))^2} \lim_{r\downarrow 0} \frac{1}{2r} \int_{(-r,r)} s\, \lambda^1(ds)\\
&= \sqrt{1+(f'(x))^2}\\
&= \left|\det D\tilde\Phi(x,0)\right|.
\end{aligned}
$$

(vii) We have

$$
\begin{aligned}
&\frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{C(r)}(x,y)\,\lambda^2(dx,dy)\\
&= \frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{\tilde\Phi(\Phi^{-1}(C)\times(-r,r))}(x,y)\,\lambda^2(dx,dy)\\
&= \frac{1}{2r}\int_{\mathbb{R}^2} \mathbb{1}_{\Phi^{-1}(C)\times(-r,r)}(z,s)\left|\det D\tilde\Phi(z,s)\right|\lambda^2(dz,ds) && \text{(Cor. 15.8)}\\
&= \int_{\mathbb{R}} \mathbb{1}_{\Phi^{-1}(C)}(z) \underbrace{\left[\frac{1}{2r}\int_{(-r,r)} \left|\det D\tilde\Phi(z,s)\right|\lambda^1(ds)\right]}_{\xrightarrow{r\downarrow 0} |\det D\tilde\Phi(z,0)|} \lambda^1(dz). && \text{(Tonelli)}
\end{aligned}
$$

Since $\Phi^{-1}(C)$ is a bounded subset of $\mathbb{R}$, we can use the result of part (vii) and dominated convergence and the proof is finished.

(viii) This follows from (i)–(iii) and the fact that

$$\left|\det D\tilde\Phi(x,0)\right| = \sqrt{1+(f'(x))^2}$$

and the geometrical meaning of the weighted area $\frac{1}{2r}\,\lambda^2(C(r))$—recall that $C(r)$ was a tubular neighbourhood of the graph.

Problem 15.7 (i) $|\det D\Phi(x)|$ is positive and measurable, hence a density and, by Lemma 10.8, $|\det D\Phi|\cdot\lambda^d$ is a measure. Therefore, $\Phi(|\det D\Phi|\cdot\lambda^d)$ is an image measure in the sense of Definition 7.7.

Using the rules for densities and integrals w.r.t. image measures we get (cf. e.g. Theorem 14.1 and/or Problem 14.1)

$$\int_M u\,d\lambda_M = \int_M u\,d\Phi\big(|\det D\Phi|\cdot\lambda^d\big) = \int_{\Phi^{-1}(M)} u\circ\Phi\cdot|\det D\Phi|\,d\lambda^d.$$

(ii) This is the formula from part (i) with $\Phi = \theta_r$; observe that $\theta_r(\mathbb{R}^n) = \mathbb{R}^n$.

(iii) The equality

$$\int u\,d\lambda^n = \int_{(0,\infty)}\int_{\{\|x\|=1\}} u(r\,x)\,r^{n-1}\,\sigma(dx)\,\lambda^1(dr)$$

is just Theorem 15.13. The equality

$$\int_{(0,\infty)} \int_{\{\|x\|=r\}} u(x)\,\sigma(dx)\,\lambda^1(dr) = \int_{(0,\infty)} \int_{\{\|x\|=1\}} u(r\,x)\,r^{n-1}\,\sigma(dx)\,\lambda^1(dr)$$

follows from part (ii).

Problem 15.8 We have

$$\Gamma\big(\tfrac{1}{2}\big) = \int_{(0,\infty)} y^{-1/2}\,e^{-y}\,\lambda(dy).$$

Using the change of variables $y = \phi(x) = x^2$, we get $D\phi(x) = 2x$ and

$$\Gamma\big(\tfrac{1}{2}\big) = 2\int_{(0,\infty)} e^{-x^2}\,\lambda(dx) = 2\int_{(-\infty,\infty)} e^{-x^2}\,\lambda(dx) \overset{(15.17)}{=} \sqrt{\pi}\,.$$

Problem 15.9 Write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Then

$$D\Phi(r,\theta,\omega) = \begin{pmatrix} \frac{\partial\Phi_1}{\partial r} & \frac{\partial\Phi_1}{\partial\theta} & \frac{\partial\Phi_1}{\partial\omega} \\ \frac{\partial\Phi_2}{\partial r} & \frac{\partial\Phi_2}{\partial\theta} & \frac{\partial\Phi_2}{\partial\omega} \\ \frac{\partial\Phi_3}{\partial r} & \frac{\partial\Phi_3}{\partial\theta} & \frac{\partial\Phi_3}{\partial\omega} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\omega & -r\sin\theta\cos\omega & -r\cos\theta\sin\omega \\ \sin\theta\cos\omega & r\cos\theta\cos\omega & -r\sin\theta\sin\omega \\ \sin\omega & 0 & r\cos\omega \end{pmatrix}$$

Developing according to the bottom row we calculate for the determinant

$$\begin{aligned}
&\det D\Phi(r,\theta,\omega) \\
&\quad= \sin\omega\det\begin{pmatrix} -r\sin\theta\cos\omega & -r\cos\theta\sin\omega \\ r\cos\theta\cos\omega & -r\sin\theta\sin\omega \end{pmatrix} \\
&\qquad + r\cos\omega\det\begin{pmatrix} \cos\theta\cos\omega & -r\sin\theta\cos\omega \\ \sin\theta\cos\omega & r\cos\theta\cos\omega \end{pmatrix} \\
&\quad= \sin\omega\Big(r^2\sin^2\theta\cos\omega\sin\omega + r^2\cos^2\theta\cos\omega\sin\omega\Big) \\
&\qquad + r\cos\omega\Big(r\cos^2\theta\cos^2\omega + r\sin^2\theta\cos^2\omega\Big) \\
&\quad= r^2\sin^2\omega\cos\omega + r^2\cos\omega\cos^2\omega \\
&\quad= r^2\cos\omega
\end{aligned}$$

where we used repeatedly the elementary relation $\sin^2\phi + \cos^2\phi = 1$.

Thus,

$$\begin{aligned}
&\iiint_{\mathbb{R}^3} u(x,y,z)\,d\lambda^3(x,y,z) \\
&\quad= \iiint_{\Phi^{-1}(\mathbb{R}^3)} u\circ\Phi(r,\theta,\omega)\,|\det D\Phi(r,\theta,\omega)|\,d\lambda^3(r,\theta,\omega) \\
&\quad= \int_0^\infty\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} U(r\cos\theta\cos\omega, r\sin\theta\cos\omega, r\sin\omega)\,r^2\cos\omega\,dr\,d\theta\,d\omega.
\end{aligned}$$

Problem 15.10 We introduce planar polar coordinates as in Example 15.11:

$$(x,y) = (r\cos\theta, r\sin\theta), \quad r > 0,\ \theta \in [0,2\pi).$$

Thus,

$$\begin{aligned}
\iint\limits_{\|x\|^2+\|y\|^2<1} & x^m\, y^n\, d\lambda^2(x,y) \\
&= \int_0^1 \int_0^{2\pi} r^{n+m+1} \cos^m\theta\, \sin^n\theta\, dr\, d\theta \\
&= \left(\int_0^1 r^{n+m+1}\, dr\right)\left(\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta\right) \qquad (*) \\
&= \frac{r^{m+n+2}}{m+n+2}\bigg|_{r=0}^{r=1}\left(\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta\right) \\
&= \frac{1}{m+n+2}\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta\,.
\end{aligned}$$

Consider the integral

$$\frac{1}{m+n+2}\int_0^{2\pi} \cos^m\theta\, \sin^n\theta\, d\theta;$$

Since sine and cosine are periodic and since we integrate over a whole period, we can also write

$$\frac{1}{m+n+2}\int_{-\pi}^{\pi} \cos^m\theta\, \sin^n\theta\, d\theta;$$

If n is odd, $\sin^n\theta$ is odd while $\cos^m\theta$ is always even. Thus, the integral equals, for odd n, zero.

Since the l.h.s. of the expression (*) is symmetric in m and n, so is the r.h.s. and we get

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\, y^n\, d\lambda^2(x,y) = 0$$

whenever m or n or both are odd.

If both m and n are even, we get

$$\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\, y^n\, d\lambda^2(x,y) = \iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ \pm x>0,\ \pm y>0}} x^m\, y^n\, d\lambda^2(x,y)$$

for any choice of signs, thus

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\, y^n\, d\lambda^2(x,y) = 4\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\, y^n\, d\lambda^2(x,y).$$

Introducing planar polar coordinates yields, as seen above, for even m and n,

$$\begin{aligned}
4\iint\limits_{\substack{\|x\|^2+\|y\|^2<1\\ x>0,\ y>0}} x^m\, y^n\, d\lambda^2(x,y) &= \frac{4}{m+n+2}\int_0^{\pi/2} \cos^m\theta\, \sin^n\theta\, d\theta \\
&= \frac{4}{m+n+2}\int_0^1 (1-t^2)^{\frac{m-1}{2}}(t^2)^{\frac{n-1}{2}}\, t\, dt
\end{aligned}$$

where we used the substitution $t = \sin\theta$ and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-t^2}$. A further substitution $s = t^2$ yields

$$\begin{aligned}
&= \frac{2}{m+n+2}\int_0^1 (1-s)^{\frac{m-1}{2}} s^{\frac{n-1}{2}}\,ds\\
&= \frac{2}{m+n+2}\int_0^1 (1-s)^{\frac{m+1}{2}-1} s^{\frac{n+1}{2}-1}\,ds\\
&= \frac{2}{m+n+2}\,B\big(\tfrac{m+1}{2},\tfrac{n+1}{2}\big)
\end{aligned}$$

which is Euler's Beta function. There is a well-known relation between the Euler Beta- and Gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{*}$$

so that, finally,

$$\iint\limits_{\|x\|^2+\|y\|^2<1} x^m\,y^n\,d\lambda^2(x,y) = \begin{cases} 0 & m \text{ or } n \text{ odd;}\\ \frac{2}{m+n+2}\frac{\Gamma\big(\frac{m+1}{2}\big)\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+m+2}{2}\big)} & \text{else}\\ \quad = \frac{\Gamma\big(\frac{m+1}{2}\big)\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+m+4}{2}\big)} \end{cases}$$

where we also used the rule that $x\Gamma(x) = \Gamma(x+1)$.

Let us briefly sketch the proof of (*): our calculation shows that

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta;$$

multiplying this formula with $r^{2x+2y-1}\,e^{-r^2}$, integrating w.r.t. r over $(0,\infty)$ and changing variables according to $s = r^2$ yields on the one hand

$$\begin{aligned}
\int_0^\infty B(x,y)\,r^{2x+2y-1}\,e^{-r^2}\,dr &= \frac{1}{2}\int_0^\infty B(x,y)\,s^{x+y-1}\,e^{-s}\,ds\\
&= \frac{1}{2}\,B(x,y)\,\Gamma(x+y)
\end{aligned}$$

while, on the other hand, we get by switching from polar to cartesian coordinates,

$$\begin{aligned}
&\int_0^\infty B(x,y)\,r^{2x+2y-1}\,e^{-r^2}\,dr\\
&\quad= 2\int_0^\infty\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\,r^{2x+2y-1}\,e^{-r^2}\,dr\,d\theta\\
&\quad= 2\int_0^\infty\int_0^{\pi/2} (r\sin\theta)^{2x-1}(r\cos\theta)^{2y-1}\,e^{-r^2}\,r\,dr\,d\theta\\
&\quad= 2\iint_{(0,\infty)\times(0,\infty)} \xi^{2x-1}\,\eta^{2y-1}\,e^{-\xi^2-\eta^2}\,d\xi\,d\eta\\
&\quad= 2\int_{(0,\infty)} \xi^{2x-1}\,e^{-\xi^2}\,d\xi\int_{(0,\infty)} \eta^{2y-1}\,e^{-\eta^2}\,d\eta\\
&\quad= \frac{1}{2}\int_{(0,\infty)} s^{x-1}\,e^{-s}\,ds\int_{(0,\infty)} t^{y-1}\,e^{-t}\,dt\\
&\quad= \frac{1}{2}\Gamma(x)\Gamma(y)
\end{aligned}$$

with the obvious applications of Tonelli's theorem and, in the penultimate equality, the obvious substitutions.

16 Uniform integrability and Vitali's convergence theorem Solutions to Problems 16.1–16.14

Problem 16.1 First, observe that

$$\lim_j u_j(x) = 0 \iff \lim_j |u_j(x)| = 0.$$

Thus,

$$\begin{aligned}
x \in \{\lim_j u_j = 0\} &\iff \forall\,\epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad \forall\, j \geqslant N_\epsilon \,:\, |u_j(x)| \leqslant \epsilon \\
&\iff \forall\,\epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad :\, \sup_{j\geqslant N_\epsilon} |u_j(x)| \leqslant \epsilon \\
&\iff \forall\,\epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad :\, x \in \{\sup_{j\geqslant N_\epsilon} |u_j| \leqslant \epsilon\} \\
&\iff \forall\,\epsilon > 0 \,:\, x \in \bigcup_{N\in\mathbb{N}} \{\sup_{j\geqslant N} |u_j| \leqslant \epsilon\} \\
&\iff \forall\, k \in \mathbb{N} \,:\, x \in \bigcup_{N\in\mathbb{N}} \{\sup_{j\geqslant N} |u_j| \leqslant 1/k\} \\
&\iff x \in \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \{\sup_{j\geqslant N} |u_j| \leqslant 1/k\}.
\end{aligned}$$

Equivalently,

$$\{\lim_j u_j = 0\}^c = \bigcup_{k\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \{\sup_{j\geqslant N} |u_j| > 1/k\}.$$

By assumption and the continuity of measures,

$$\mu\Big(\bigcap_{N\in\mathbb{N}} \{\sup_{j\geqslant N} |u_j| > 1/k\}\Big) = \lim_N \mu\Big(\{\sup_{j\geqslant N} |u_j| > 1/k\}\Big) = 0$$

and, since countable unions of null sets are again null sets, we conclude that

$$\{\lim_j u_j = 0\} \quad \text{has full measure.}$$

Problem 16.2 Note that

$$\begin{aligned}
x \in \Big\{\sup_{j\geqslant k} |u_j| > \epsilon\Big\} &\iff \sup_{j\geqslant k} |u_j(x)| > \epsilon \\
&\iff \exists\, j \geqslant k \,:\, |u_j(x)| > \epsilon \\
&\iff x \in \bigcup_{j\geqslant k} \{|u_j| > \epsilon\}
\end{aligned}$$

and since

$$\bigcup_{j\geqslant k}\{|u_j|>\epsilon\}\downarrow\bigcap_{k\in\mathbb{N}}\bigcup_{j\geqslant k}\{|u_j|>\epsilon\}\overset{\text{def}}{=}\limsup_{j\to\infty}\{|u_j|>\epsilon\}$$

we can use the continuity of measures to get

$$\lim_k \mu\Big(\sup_{j\geqslant k}|u_j|>\epsilon\Big)=\lim_k \mu\Big(\bigcup_{j\geqslant k}\{|u_j|>\epsilon\}\Big)=\mu\Big(\bigcap_{k\in\mathbb{N}}\bigcup_{j\geqslant k}\{|u_j|>\epsilon\}\Big).$$

This, and the result of Problem 16.1 show that either of the following two equivalent conditions

$$\lim_{k\to\infty}\mu\Big(\sup_{j\geqslant k}|u_j|\geqslant\epsilon\Big)=0\qquad\forall\,\epsilon>0;$$
$$\mu\Big(\limsup_{j\to\infty}\{|u_j|\geqslant\epsilon\}\Big)=0\qquad\forall\,\epsilon>0;$$

ensure the almost everywhere convergence of $\lim_j u_j(x)=0$.

Problem 16.3 • Assume first that $u_j\to u$ in μ-measure, that is,

$$\forall\,\epsilon>0,\quad\forall\,A\in\mathcal{A},\ \mu(A)<\infty\ :\ \lim_j\mu\big(\{|u_j-u|>\epsilon\}\cap A\big)=0.$$

Since

$$|u_j-u_k|\leqslant|u_j-u|+|u-u_k|\quad\forall j,k\in\mathbb{N}$$

we see that

$$\{|u_j-u_k|>2\epsilon\}\subset\{|u_j-u|>\epsilon\}\cup\{|u-u_k|>\epsilon\}$$

(since, otherwise $|u_j-u_k|\leqslant\epsilon+\epsilon=2\epsilon$). Thus, we get for every measurable set A with finite μ-measure that

$$\begin{aligned}&\mu\big(\{|u_j-u_k|>2\epsilon\}\cap A\big)\\&\quad\leqslant\mu\big[(\{|u_j-u|>\epsilon\}\cap A)\cup(\{|u_k-u|>\epsilon\}\cap A)\big]\\&\quad\leqslant\mu\big[\{|u_j-u|>\epsilon\}\cap A\big]+\mu\big[\{|u_k-u|>\epsilon\}\cap A\big]\end{aligned}$$

and each of these terms tend to infinity as $j,k\to\infty$.

• Assume now that $|u_j-u_k|\to0$ in μ-measure as $j,k\to\infty$. Let $(A_\ell)_\ell$ be an exhausting sequence such that $A_\ell\uparrow X$ and $\mu(A_j)<\infty$.

The problem is to identify the limiting function.

Fix ℓ. By assumption, we can choose $N_j\in\mathbb{N}$, $j\in\mathbb{N}$, such that

$$\forall\,m,n\geqslant N_j\ :\ \mu\big(\{|u_m-u_n|>2^{-j}\}\cap A_\ell\big)<2^{-j}.$$

(Note that N_j may depend on ℓ, but we suppress this dependency as ℓ is fixed.) By enlarging N_j, if needed, we can always assume that

$$N_1<N_2<\cdots<N_j<N_{j+1}\to\infty.$$

Consequently, there is an exceptional set $E_j \subset A_\ell$ with $\mu(E_j \cap A_\ell) < 2^{-j}$ such that

$$|u_{N_{j+1}}(x) - u_{N_j}(x)| \leqslant 2^{-j} \quad \forall\, x \in A_\ell \setminus E_j$$

and, if $E_i^* := \bigcup_{j \geqslant i} E_j$ we have $\mu(E_i \cap A_\ell) \leqslant 2 \cdot 2^{-i}$ as well as

$$|u_{N_{j+1}}(x) - u_{N_j}(x)| \leqslant 2^{-j} \quad \forall\, j \geqslant i, \quad \forall\, x \in A_\ell \setminus E_i^*.$$

This means that

$$\sum_j (u_{N_{j+1}} - u_{N_j}) \text{ converges uniformly for } x \in A_\ell \setminus E_i^*$$

so that $\lim_j u_{N_j}$ exists uniformly on $A_\ell \setminus E_i^*$ for all i. Since $\mu(E_i^* \cap A_\ell) < 2 \cdot 2^{-i}$ we conclude that

$$\lim_j u_{N_j} \mathbb{1}_{A_\ell} = u^{(\ell)} \mathbb{1}_{A_\ell} \text{ exists almost everywhere}$$

for some $u^{(\ell)}$. Since, however, a.e. limits are unique (up to a null set, that is) we know that $u^{(\ell)} = u^{(m)}$ a.e. on $A_\ell \cap A_m$ so that there is a (up to null sets) unique limit function u satisfying

$$\lim_j u_{N_j} = u \text{ exists a.e., hence in measure by Lemma 16.4.} \tag{*}$$

Thus, we have found a candidate for the limit of our Cauchy sequence. In fact, since

$$|u_k - u| \leqslant |u_k - u_{N_j}| + |u_{N_j} - u|$$

we have

$$\begin{aligned} &\mu(\{|u_k - u| > \epsilon\} \cap A_\ell) \\ &\qquad \leqslant \mu(\{|u_k - u_{N_j}| > \epsilon\} \cap A_\ell) + \mu(\{|u_{N_j} - u| > \epsilon\} \cap A_\ell) \end{aligned}$$

and the first expression on the right-hand side tends to zero (as $k, N(j) \to \infty$) because of the assumption, while the second term tends to zero (as $N(j) \to \infty$) because of (*))

Problem 16.4 (i) This sequence converges in measure to $f \equiv 0$ since for $\epsilon \in (0,1)$

$$\lambda(|f_{n,j}| > \epsilon) = \lambda[(j-1)/n, j/n] = \tfrac{1}{n} \xrightarrow{n\to\infty} 0.$$

This means, however, that potential a.e. and $\mathcal{L}^p$-limits must be $f \equiv 0$, too. Since for every x

$$\liminf f_{n,j}(x) = 0 < \infty = \limsup f_{n,j}$$

the sequence cannot converge at any point.

Also the $\mathcal{L}^p$-limit (if $p \geqslant 1$) does not exist, since

$$\int |f_{n,j}|^p \, d\lambda = n^p \lambda[(j-1)/n, j/n] = n^{p-1}.$$

(ii) As in (i) we see that $g_n \xrightarrow{\mu} g \equiv 0$. Similarly,

$$\int |g_n|^p \, d\mu = n^p \lambda(0, 1/n) = n^{p-1}$$

so that the $\mathcal{L}^p$-limit does not exist. The pointwise limit, however, exists since

$$\lim_{n\to\infty} n\mathbb{1}_{(0,n)}(x) = 0.$$

for every $x \in (0,1)$.

(iii) The shape of g_n is that of a triangle with base $[0, 1/n]$. Thus, for every $\epsilon > 0$,

$$\lambda(|h_n| > \epsilon) \leqslant \lambda[0, 1/n] = \tfrac{1}{n}$$

which shows that $h_n \xrightarrow{\mu} h \equiv 0$. This must be, if the respective limits exist, also the limiting function for a.e. and $\mathcal{L}^p$-convergence. Since

$$\int |h_n|^p \, d\lambda = a_n^p \, \tfrac{1}{2} \, \lambda[0, 1/n] = \frac{a_n^p}{2n}$$

we have $\mathcal{L}^p$-convergence if, and only if, the sequence a_n^p/n tends to zero as $n \to \infty$.

We have, however, always a.e. convergence since the support of the function h_n is $[0, 1/n]$ and this shrinks to $\{0\}$ which is a null set. Thus,

$$\lim_n a_n(1 - nx)^+ = 0$$

except, possibly, at $x = 0$.

Problem 16.5 We claim that

(i) $au_j + bw_j \to au + bw$;

(ii) $\max(u_j, w_j) \to \max(u, w)$;

(iii) $\min(u_j, w_j) \to \min(u, w)$;

(iv) $|u_j| \to |u|$.

Note that

$$|au_j + bw_j - au - bw| \leqslant |a||u_j - u| + |b||w_j - w|$$

so that

$$\{|au_j + bw_j - au - bw| > 2\epsilon\} \subset \{|u_j - u| > \epsilon/|a|\} \cup \{|w_j - w| > \epsilon/|b|\}.$$

This proves the first limit.

Since, by the lower triangle inequality,

$$||u_j| - |u|| \leqslant |u_j - u|$$

we get

$$\{||u_j| - |u|| > \epsilon\} \subset \{|u_j - u| > \epsilon\}$$

and $|u_j| \to |u|$ follows.

Finally, since

$$\max u_j, w_j = \frac{1}{2}\big(u_j + w_j + |u_j - w_j|\big)$$

we get $\max u_j, w_j \to \max u, w$ by using rules (i) and (iv) several times. The minimum is treated similarly.

Problem 16.6 The hint is somewhat misleading since this construction is not always possible (or sensible). Just imagine $\mathbb{R}$ with the counting measure. Then $X_{\sigma f}$ would be all of $\mathbb{R}$... What I had in mind when giving this hint was a construction along the following lines:

Consider Lebesgue measure λ in $\mathbb{R}$ and define $f := \mathbb{1}_F + \infty\mathbb{1}_{F^c}$ where $F = [-1, 1]$ (or any other set of finite Lebesgue measure). Then $\mu := f \cdot \lambda$ is a not σ-finite measure. Moreover, Take any sequence $u_n \xrightarrow{\lambda} u$ converging in λ-measure. Then

$$\mu(\{|u_n - u| > \epsilon\} \cap A) = \lambda(\{|u_n - u| > \epsilon\} \cap A)$$

since all sets A with $\mu(A) < \infty$ are contained in F and $\lambda(F) = \mu(F) < \infty$. Thus, $u_n \xrightarrow{\mu} u$. However, changing u *arbitrarily* on $\mathbb{1}_{F^c}$ also yields a limit point in μ-measure since, as mentioned above, all sets of finite μ-measure are within F.

This pathology cannot happen in a σ-finite measure space, cf. Lemma 16.5.

Problem 16.7 (i) Fix $\epsilon > 0$. Then

$$\begin{aligned}\int |u - u_j|\,d\mu &= \int_A |u - u_j|\,d\mu \\ &= \int_{A\cap\{|u-u_j|\leqslant\epsilon\}} |u - u_j|\,d\mu + \int_{A\cap\{|u-u_j|>\epsilon\}} |u - u_j|\,d\mu \\ &\leqslant \int_{A\cap\{|u-u_j|\leqslant\epsilon\}} \epsilon\,d\mu + \int_{A\cap\{|u-u_j|>\epsilon\}} (|u| + |u_j|)\,d\mu \\ &\leqslant \epsilon\mu(A) + 2C\,\mu\big(A \cap \{|u - u_j| > \epsilon\}\big) \\ &\xrightarrow{j\to\infty} \epsilon\mu(A) \\ &\xrightarrow{\epsilon\to 0} 0.\end{aligned}$$

(ii) Note that u_j converges almost everywhere and in λ-measure to $u \equiv 0$. However,

$$\int |u_j|\,d\lambda = \lambda[j, j+1] = 1 \neq 0$$

so that the limit—if it exists—cannot be $u \equiv 0$. Since this is, however, the canonical candidate, we conclude that there is no $\mathcal{L}^1$ convergence.

(iii) The limit depends on the set A which is fixed. This means that we are, essentially, dealing with a finite measure space.

Problem 16.8 Mind the misprint throughout the problem: $\mu = P$ so that ρ_μ, g_μ, d_μ should read ρ_P, g_P, d_P.

A pseudo-metric is symmetric (d_2) and satisfies the triangle inequality (d_3).

(i) First we note that $\rho_\mu(X,Y) \in [0,1]$ is well-defined. That it is symmetric (d_2) is obvious. For the triangle inequality we observe that for three random variables X,Y,Z and numbers $\epsilon,\delta > 0$ we have

$$|X-Z| \leqslant |X-Y| + |Y-Z|$$

implying that

$$\{|X-Z| > \epsilon+\delta\} \subset \{|X-Y| > \epsilon\} \cup \{|Y-Z| > \delta\}$$

so that

$$P(|X-Z| > \epsilon+\delta) \leqslant P(|X-Y| > \epsilon) + P(|Y-Z| > \delta).$$

If $\epsilon > \rho_P(X,Y)$ and $\delta > \rho_P(Y,Z)$ we find

$$P(|X-Z| > \epsilon+\delta) \leqslant P(|X-Y| > \epsilon) + P(|Y-Z| > \delta) \leqslant \epsilon+\delta$$

which means that

$$\rho_P(X,Z) \leqslant \epsilon+\delta.$$

Passing to the infimum of all possible δ- and ϵ-values we get

$$\rho_P(X,Z) \leqslant \rho_P(X,Y) + \rho_P(Y,Z).$$

(ii) Assume first that $\rho_P(X_j,X) \xrightarrow{j\to\infty} 0$. Then

$$\begin{aligned}\rho_P(X_j,X) \xrightarrow{j\to\infty} 0 &\iff \exists\,(\epsilon_j)_j \subset \mathbb{R}_+ \,:\, P(|X-X_j| > \epsilon_j) \leqslant \epsilon_j\\ &\implies \forall\,\epsilon > \epsilon_j \,:\, P(|X-X_j| > \epsilon) \leqslant \epsilon_j.\end{aligned}$$

Thus, for given $\epsilon > 0$ we pick $N = N(\epsilon)$ such that $\epsilon > \epsilon_j$ for all $j \geqslant N$ (possible as $\epsilon_j \to 0$). Then we find

$$\forall\,\epsilon > 0\;\exists\,N(\epsilon) \in \mathbb{N}\;\forall\,j \geqslant N(\epsilon) \,:\, P(|X-X_j| > \epsilon) \leqslant \epsilon_j;$$

this means, however, that $P(|X-X_j| > \epsilon) \xrightarrow{j\to\infty} 0$ for any choice of $\epsilon > 0$.

Conversely, assume that $X_j \xrightarrow{P} 0$. Then

$$\begin{aligned}&\forall\,\epsilon > 0 \,:\, \lim_j P(|X-X_j| > \epsilon) = 0\\ &\iff \forall\,\epsilon,\delta > 0\;\exists\,N(\epsilon,\delta)\;\forall\,j \geqslant N(\epsilon,\delta) \,:\, P(|X-X_j| > \epsilon) < \delta\\ &\implies \forall\,\epsilon > 0\;\exists\,N(\epsilon)\;\forall\,j \geqslant N(\epsilon) \,:\, P(|X-X_j| > \epsilon) < \epsilon\\ &\implies \forall\,\epsilon > 0\;\exists\,N(\epsilon)\;\forall\,j \geqslant N(\epsilon) \,:\, \rho_P(X,X_j) \leqslant \epsilon\\ &\implies \lim_j \rho_P(X,X_j) = 0.\end{aligned}$$

(iii) We have

$$\begin{aligned}\rho(X_j,X_k) \xrightarrow{j,k\to\infty} 0 &\overset{\text{(ii)}}{\iff} X_j - X_k \xrightarrow[j,k\to\infty]{P} 0\\ &\overset{\text{P16.3}}{\iff} \exists\,X \,:\, X_k \xrightarrow[k\to\infty]{P} X\\ &\overset{\text{(ii)}}{\iff} \exists\,X \,:\, \rho(X,X_k) \xrightarrow{k\to\infty} 0\end{aligned}$$

(iv) Note that for $x, y > 0$

$$\frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y} \leqslant \frac{x}{1+x} + \frac{y}{1+y}$$

and

$$(x+y)\wedge 1 = \begin{cases} x+y = (x\wedge 1) + (y \wedge 1) & \text{if } x+y \leqslant 1; \\ 1 \leqslant (x\wedge 1) + (y\wedge 1) & \text{if } x+y \geqslant 1. \end{cases}$$

This means that both g_P and d_P satisfy the triangle inequality, that is (d_3). Symmetry, i.e. (d_2), is obvious.

Moreover, since for all $x \geqslant 0$

$$\frac{x}{1+x} \leqslant x \wedge 1 \leqslant 2\,\frac{x}{1+x}$$

(consider the cases $x \leqslant 1$ and $x \geqslant 1$ separately), we have

$$g_P(X,Y) \leqslant d_P(X,Y) \leqslant 2g_P(X,Y)$$

which shows that g_P and d_P have the same Cauchy sequences. Moreover, for all $\epsilon \leqslant 1$,

$$\begin{aligned} P(|X-Y| > \epsilon) &= P(|X-Y| \wedge 1 > \epsilon) \\ &\leqslant \frac{1}{\epsilon} \int |X-Y| \wedge 1\, dP \\ &= \frac{1}{\epsilon}\, d_P(X,Y) \end{aligned}$$

so that (because of (iii)) any d_P Cauchy sequence is a ρ_P Cauchy sequence. And since for all $\epsilon \leqslant 1$ also

$$\begin{aligned} d_P(X,Y) &= \int_{|X-Y|>\epsilon} |X-Y| \wedge 1\, dP + \int_{|X-Y|\leqslant\epsilon} |X-Y| \wedge 1\, dP \\ &\leqslant \int_{|X-Y|>\epsilon} 1\, dP + \int_{|X-Y|\leqslant\epsilon} \epsilon\, dP \\ &\leqslant P(|X-Y|>\epsilon) + \epsilon, \end{aligned}$$

all ρ_P Cauchy sequences are d_P Cauchy sequences, too.

Problem 16.9 Note that the sets A_j are of finite μ-measure. Observe that the functions $f_j := u\mathbb{1}_{A_j}$

- converge in μ-measure to $f \equiv 0$:

$$\mu(\{|f_j| > \epsilon\} \cap A_j) \leqslant \mu(A_j) \xrightarrow{j\to\infty} 0.$$

- are uniformly integrable:

$$\sup_j \int_{\{|f_j|>|u|\}} |f_j|\, d\mu = 0$$

since $|f_j| = |u\mathbb{1}_{A_j}| \leqslant |u|$ and $|u|$ is integrable.

Therefore, Vitali's Theorem shows that $f_j \to 0$ in $\mathcal{L}^1$ so that $\int f_j \, d\mu = \int_{A_j} u \, d\mu \to 0$.

Problem 16.10 (i) Trivial. More interesting is the assertion that

A sequence $(x_n)_n \subset \mathbb{R}$ converges to 0 if, and only if, every subsequence $(x_{n_k})_k$ contains some sub-subsequence $(\tilde{x}_{n_k})_k$ which converges to 0.

Necessity is again trivial. Sufficiency: assume that $(x_n)_n$ does not converge to 0. Then the sequence $(\min\{|x_n|, 1\})_n$ is bounded and still does not converge to 0. Since this sequence is bounded, it contains a convergent subsequence $(x_{n_k})_k$ with some limit $\alpha \neq 0$. But then $(x_{n_k})_k$ cannot contain a sub-subsequence $(\tilde{x}_{n_k})_k$ which is a null sequence.

(ii) If $u_n \xrightarrow{\mu} u$, then every subsequence $u_{n_k} \xrightarrow{\mu} u$. Thus, using the argument from the proof of Problem 16.3 we can extract a sub-subsequence $(\tilde{u}_{n_k})_k \subset (u_{n_k})_k$ such that

$$\lim_k \tilde{u}_{n_k}(x)\mathbb{1}_A(x))u(x)\mathbb{1}_A(x) \text{ almost everywhere.} \tag{*}$$

Note that (unless we are in a σ-finite measure space) the exceptional set may depend on the testing set A.

Conversely, assume that every subsequence $(u_{n_k})_k \subset (u_n)_n$ has a sub-subsequence $(\tilde{u}_{n_k})_k$ satisfying (*). Because of Lemma 16.4 we have

$$\lim_k \mu\big(\{|\tilde{u}_{n_k} - u| > \epsilon\} \cap A\big) = 0.$$

Assume now that u_n does not converge in μ-measure on A to u. Then

$$x_n := \mu(\{|u_n - u| > \epsilon\} \cap A) \not\to 0.$$

Since the whole sequence $(x_n)_n$ is bounded (by $\mu(A)$) there exists some subsequence $(x_{n_k})_k$ given by $(u_{n_k})_k$ such that

$$x_{n_k} = \mu(\{|u_{n_k} - u| > \epsilon\} \cap A) \to \alpha \neq 0.$$

This contradicts, however, the fact that x_{n_k} has itself a subsequence converging to zero.

(iii) Fix some set A of finite μ-measure. All conclusions below take place relative to resp. on this set only.

If $u_n \xrightarrow{\mu} u$ we have for every subsequence $(u_{n_k})_k$ a sub-subsequence $(\tilde{u}_{n_k})_k$ with $\tilde{u}_{n_k} \to u$ a.e. Since Φ is continuous, we get $\Phi \circ \tilde{u}_{n_k} \to \Phi \circ u$ a.e.

This means, however, that every subsequence $(\Phi \circ u_{n_k})_k$ of $(\Phi \circ u_n)_n$ has a sub-subsequence $(\Phi \circ \tilde{u}_{n_k})_k$ which converges a.e. to $\Phi \circ u$. Thus, part (ii) says that $\Phi \circ u_n \xrightarrow{\mu} \Phi \circ u$.

Problem 16.11 Since $\mathcal{F}$ and $\mathcal{G}$ are uniformly integrable, we find for any given $\epsilon > 0$ functions $f_\epsilon, g_\epsilon \in \mathcal{L}^1_+$ such that

$$\sup_{f\in\mathcal{F}} \int_{\{|f|>f_\epsilon\}} |f|\,d\mu \leqslant \epsilon \quad\text{and}\quad \sup_{g\in\mathcal{G}} \int_{\{|g|>g_\epsilon\}} |g|\,d\mu \leqslant \epsilon.$$

We will use this notation throughout.

(i) Since $f := |f_1| + \cdots + |f_n| \in \mathcal{L}^1_+$ we find that

$$\int_{\{|f_j|>f\}} |f_j|\,d\mu = \int_\varnothing |f_j|\,d\mu = 0$$

uniformly for all $1 \leqslant j \leqslant n$. This proves uniform integrability.

(ii) Instead of $\{f_1, \dots, f_N\}$ (which is uniformly integrable because of (i)) we show that $\mathcal{F} \cup \mathcal{G}$ is uniformly integrable.

Set $h_\epsilon := f_\epsilon + g_\epsilon$. Then $h_\epsilon \in \mathcal{L}^1_+$ and

$$\{|w| \geqslant f_\epsilon + g_\epsilon\} \subset \{|w| \geqslant f_\epsilon\} \cap \{|w| \geqslant g_\epsilon\}$$

which means that we have

$$\int_{\{|w|>h_\epsilon\}} |w|\,d\mu \leqslant \begin{cases} \int_{\{|w|>f_\epsilon\}} |w|\,d\mu \leqslant \epsilon & \text{if } \ w \in \mathcal{F} \\ \int_{\{|w|>g_\epsilon\}} |w|\,d\mu \leqslant \epsilon & \text{if } \ w \in \mathcal{G}. \end{cases}$$

Since this is uniform for all $w \in \mathcal{F} \cup \mathcal{G}$, the claim follows.

(iii) Set $h_\epsilon := f_\epsilon + g_\epsilon \in \mathcal{L}^1_+$. Since $|f + g| \leqslant |f| + |g|$ we have

$$\begin{aligned} \{|f+g| > h_\epsilon\} &\subset \{|f| > h_\epsilon\} \cup \{|g| > h_\epsilon\} \\ &= \big[\{|f| > h_\epsilon\} \cap \{|g| > h_\epsilon\}\big] \\ &\qquad \cup \big[\{|f| > h_\epsilon\} \cap \{|g| \leqslant h_\epsilon\}\big] \\ &\qquad \cup \big[\{|f| \leqslant h_\epsilon\} \cap \{|g| > h_\epsilon\}\big] \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{\{|f+g|>h_\epsilon\}} |f+g|\,d\mu \\ &\leqslant \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} \big(|f|+|g|\big)\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|\leqslant h_\epsilon\}}} |f| \vee |g|\,d\mu + \int_{\substack{\{|f|\leqslant h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |f| \vee |g|\,d\mu \\ &= \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |f|\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |g|\,d\mu + \int_{\substack{\{|f|>h_\epsilon\}\\ \cap\{|g|\leqslant h_\epsilon\}}} |f|\,d\mu + \int_{\substack{\{|f|\leqslant h_\epsilon\}\\ \cap\{|g|>h_\epsilon\}}} |g|\,d\mu \\ &\leqslant \int_{\{|f|>h_\epsilon\}} |f|\,d\mu + \int_{\{|g|>h_\epsilon\}} |g|\,d\mu + \int_{\{|f|>h_\epsilon\}} |f|\,d\mu + \int_{\{|g|>h_\epsilon\}} |g|\,d\mu \\ &\leqslant \int_{\{|f|>f_\epsilon\}} |f|\,d\mu + \int_{\{|g|>g_\epsilon\}} |g|\,d\mu + \int_{\{|f|>f_\epsilon\}} |f|\,d\mu + \int_{\{|g|>g_\epsilon\}} |g|\,d\mu \\ &\leqslant 4\epsilon \end{aligned}$$

uniformly for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

(iv) This follows from (iii) if we set

- $t\mathcal{F} \rightsquigarrow \mathcal{F}$,
- $(1-t)\mathcal{F} \rightsquigarrow \mathcal{G}$,
- $tf_\epsilon \rightsquigarrow f_\epsilon$,
- $(1-t)f_\epsilon \rightsquigarrow g_\epsilon$,

and observe that the calculation is uniform for all $t \in [0,1]$.

(v) Without loss of generality we can assume that $\mathcal{F}$ is convex, i.e. coincides with its convex hull.

Let u be an element of the $\mathcal{L}^1$-closure of (the convex hull of) $\mathcal{F}$. Then there is a sequence

$$(f_j)_j \subset \mathcal{F} \,:\, \lim_j \|u - f_j\|_1 = 0.$$

We have, because of $|u| \leqslant |u - f_j| + |f_j|$,

$$\begin{aligned}\{|u| > f_\epsilon\} &\subset \{|u-f_j| > f_\epsilon\} \cup \{|f_j| > f_\epsilon\}\\ &= \big[\{|u-f_j| > f_\epsilon\} \cap \{|f_j| > f_\epsilon\}\big]\\ &\quad \cup \big[\{|u-f_j| > f_\epsilon\} \cap \{|f_j| \leqslant f_\epsilon\}\big]\\ &\quad \cup \big[\{|u-f_j| \leqslant f_\epsilon\} \cap \{|f_j| > f_\epsilon\}\big]\end{aligned}$$

that

$$\begin{aligned}&\int_{\{|u|>f_\epsilon\}} |u|\,d\mu\\ &\leqslant \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u|\,d\mu + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|\leqslant f_\epsilon\}}} |u|\,d\mu + \int_{\substack{\{|u-f_j|\leqslant f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u|\,d\mu\\ &\leqslant \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u-f_j|\,d\mu + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |f_j|\,d\mu\\ &\qquad + \int_{\substack{\{|u-f_j|>f_\epsilon\}\\ \cap\{|f_j|\leqslant f_\epsilon\}}} |u-f_j| \vee |f_j|\,d\mu + \int_{\substack{\{|u-f_j|\leqslant f_\epsilon\}\\ \cap\{|f_j|>f_\epsilon\}}} |u-f_j| \vee |f_j|\,d\mu\\ &\leqslant \|u-f_j\|_1 + \int_{\{|f_j|>f_\epsilon\}} |f_j|\,d\mu + \|u-f_j\|_1 + \int_{\{|f_j|>f_\epsilon\}} |f_j|\,d\mu\\ &\leqslant 2\,\|u-f_j\|_1 + 2\epsilon\\ &\xrightarrow{j\to\infty} 2\epsilon.\end{aligned}$$

Since this holds uniformly for all such u, we are done.

Problem 16.12 By assumption,

$$\forall\, \epsilon > 0\ \exists\, w_\epsilon \in \mathcal{L}^1_+ \,:\, \sup_{f\in\mathcal{F}} \int_{\{|f|>w_\epsilon\}} |f|\,d\mu \leqslant \epsilon.$$

Now observe that

$$\begin{aligned}
&\int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}} \sup_{1\leqslant j\leqslant k}|f_j|\,d\mu \\
&\leqslant \sum_{\ell=1}^{k} \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}\cap\{|f_\ell|=\sup_{1\leqslant j\leqslant k}|f_j|\}} |f_\ell|\,d\mu \\
&\leqslant \sum_{\ell=1}^{k} \int_{\{|f_\ell|>w_\epsilon\}} |f_\ell|\,d\mu \\
&\leqslant \sum_{\ell=1}^{k} \epsilon \\
&= k\,\epsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \sup_{1\leqslant j\leqslant k}|f_j|\,d\mu \\
&\leqslant \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|\leqslant w_\epsilon\}} \sup_{1\leqslant j\leqslant k}|f_j|\,d\mu + \int_{\{\sup_{1\leqslant j\leqslant k}|f_j|>w_\epsilon\}} \sup_{1\leqslant j\leqslant k}|f_j|\,d\mu \\
&\leqslant \int w_\epsilon\,d\mu + k\epsilon
\end{aligned}$$

and we get

$$\lim_{k\to\infty}\frac{1}{k}\int \sup_{1\leqslant j\leqslant k}|f_j|\,d\mu \leqslant \lim_{k\to\infty}\frac{1}{k}\int w_\epsilon\,d\mu + \epsilon = \epsilon$$

which proves our claim as $\epsilon > 0$ was arbitrary.

Problem 16.13 Since the function $u \equiv R$, $R > 0$, is integrable w.r.t. the probability measure P, we get

$$\begin{aligned}
\int_{\{|u_j|>R\}} |u_j|\,dP &\leqslant \int_{\{|u_j|>R\}} |u_j|\,\frac{|u_j|^{p-1}}{R^{p-1}}\,dP \\
&= \frac{1}{R^{p-1}}\int_{\{|u_j|>R\}} |u_j|^p\,dP \\
&\leqslant \frac{1}{R^{p-1}}\int |u_j|^p\,dP \\
&\leqslant \frac{1}{R^{p-1}}\sup_k\int |u_k|^p\,dP \\
&= \frac{1}{R^{p-1}}\sup_k\|u_k\|_p^p
\end{aligned}$$

which converges to zero as $R\to\infty$. This proves uniform integrability.

Counterexample:

Vitali's theorem implies that a counterexample should satisfy

$$u_j \xrightarrow[j\to\infty]{P} u, \qquad \|u_j\|_1 = 1, \qquad u_j \text{ does not converge in } \mathcal{L}^1.$$

Consider, for example, the probability space $((0,1),\mathcal{B}(0,1),dx)$ and the sequence

$$u_j := j\cdot\mathbb{1}_{(0,1/j)}.$$

Then $u_j \to 0$ pointwise (everywhere!), hence in measure. This is also the expected $\mathcal{L}^1$ limit, if it exists. Moreover,

$$\|u_j\|_1 = \int u_j \, dx = 1$$

which means that u_j cannot converge in $\mathcal{L}^1$ to the expected limit $u \equiv 0$, i.e. it does not converge in $\mathcal{L}^1$.

Vitali's theorem shows now that $(u_j)_j$ cannot be uniformly integrable.

We can verify this fact also directly: for $R > 0$ and all $j > R$ we get

$$\int_{\{|u_j|>R\}} |u_j| \, dx = \int u_j \, dx = 1$$

which proves

$$\sup_j \int_{\{|u_j|>R\}} |u_j| \, dx = 1 \qquad \forall \, R > 0$$

and $(u_j)_j$ cannot be uniformly integrable (in view of the equivalent characterizations of uniform integrability on finite measure spaces, cf. Theorem 16.8)

Problem 16.14 We have

$$\begin{aligned}\sum_{j=k}^{\infty} j \, \mu(j < |f| \leqslant j+1) &= \sum_{j=k}^{\infty} \int_{\{j<|f|\leqslant j+1\}} j \, d\mu \\ &\leqslant \sum_{j=k}^{\infty} \int_{\{j<|f|\leqslant j+1\}} |f| \, d\mu \\ &= \int_{\{|f|>k\}} |f| \, d\mu,\end{aligned}$$

and, since $2j \geqslant j+1$ for all $j \in\in \mathbb{N}$, also

$$\begin{aligned}2\sum_{j=k}^{\infty} j \, \mu(j < |f| \leqslant j+1) &= \sum_{j=k}^{\infty} 2j \, \mu(j < |f| \leqslant j+1) \\ &= \sum_{j=k}^{\infty} \int_{\{j<|f|\leqslant j+1\}} 2j \, d\mu \\ &\geqslant \sum_{j=k}^{\infty} \int_{\{j<|f|\leqslant j+1\}} |f| \, d\mu \\ &= \int_{\{|f|>k\}} |f| \, d\mu.\end{aligned}$$

This shows that

$$\int_{\{|f|>k\}} |f| \, d\mu \leqslant 2\sum_{j=k}^{\infty} j \, \mu(j < |f| \leqslant j+1) \leqslant 2\int_{\{|f|>k\}} |f| \, d\mu$$

and this implies

$$\sup_{f\in\mathcal{F}} \int_{\{|f|>k\}} |f| \, d\mu \simeq \sup_{f\in\mathcal{F}} \sum_{j=k}^{\infty} j \, \mu(j < |f| \leqslant j+1).$$

This proves the claim (since we are in a finite measure space where $u \equiv k$ is an integrable function!)

17 Martingales
Solutions to Problems 17.1–17.16

Problem 17.1 Since $\mathcal{A}_0 = \{\varnothing, X\}$ an $\mathcal{A}_0$-measurable function u must satisfy $\{u = s\} = \varnothing$ or $= X$, i.e. all $\mathcal{A}_0$-measurable functions are constants.

So if $(u_j)_{j\in\mathbb{N}_0}$ is a martingale, u_0 is a constant and we can calculate its value because of the martingale property:

$$\int_X u_0\,d\mu = \int_X u_1\,d\mu \implies u_0 = \mu(X)^{-1}\int_X u_1\,d\mu. \tag{*}$$

Conversely, since $\mathcal{A}_0 = \{\varnothing, X\}$ and since

$$\int_\varnothing u_0\,d\mu = \int_\varnothing u_1\,d\mu$$

always holds, it is clear that the calculation and choice in (*) is necessary and sufficient for the claim.

Problem 17.2 We consider only the martingale case, the other two cases are similar.

(a) Since $\mathcal{B}_j \subset \mathcal{A}_j$ we get

$$\int_A u_j\,d\mu = \int_A u_{j+1}\,d\mu \quad \forall\, A \in \mathcal{A}_j$$
$$\implies \int_B u_j\,d\mu = \int_B u_{j+1}\,d\mu \quad \forall\, B \in \mathcal{B}_j$$

showing that $(u_j, \mathcal{B}_j)_j$ is a martingale.

(b) It is clear that the above implication cannot hold if we enlarge $\mathcal{A}_j$ to become $\mathcal{C}_j$. Just consider the following 'extreme' case (to get a counterexample): $\mathcal{C}_j = \mathcal{A}$ for all j. Any martingale $(u_j, \mathcal{C})_j$ must satisfy,

$$\int_A u_j\,d\mu = \int_A u_{j+1}\,d\mu \quad \forall\, A \in \mathcal{A}.$$

Considering the sets $A := \{u_j < u_{j+1}\} \in \mathcal{A}$ and $A' := \{u_j > u_{j+1}\} \in \mathcal{A}$ we conclude that

$$0 = \int_{\{u_j > u_{j+1}\}} (u_j - u_{j+1})\,d\mu \implies \mu(\{u_j > u_{j+1}\}) = 0$$

and, similarly $\mu(\{u_j < u_{j+1}\}) = 0$ so that $u_j = u_{j+1}$ almost everywhere and for all j. This means that, if we start with a non-constant martingale $(u_j, \mathcal{A}_j)_j$, then this can never be a martingale w.r.t. the filtration $(\mathcal{C}_j)_j$.

Problem 17.3 For the notation etc. we refer to Problem 4.13. Since the completion $\mathcal{A}_j^*$ is given by

$$\mathcal{A}_j^* = \sigma(\mathcal{A}_j, \mathcal{N}), \quad \mathcal{N} := \{M \subset X \,:\, \exists N \in \mathcal{A},\ N \supset M,\ \mu(N) = 0\}$$

we find that for all $A_j^* \in \mathcal{A}_j^*$ there exists some $A_j \in \mathcal{A}_j$ such that

$$A_j^* \setminus A_j \cup A_j \setminus A_j^* \in \mathcal{N}.$$

Writing $\bar{\mu}$ for the unique extension of μ onto $\mathcal{A}^*$ (and thus onto $\mathcal{A}_j^*$ for all j) we get for A_j^*, A_j as above

$$\begin{aligned}\left|\int_{A_j^*} u_j\,d\bar{\mu} - \int_{A_j} u_j\,d\mu\right| &= \left|\int_{A_j^*} u_j\,d\bar{\mu} - \int_{A_j} u_j\,d\bar{\mu}\right| \\ &= \left|\int (\mathbb{1}_{A_j^*} - \mathbb{1}_{A_j}) u_j\,d\bar{\mu}\right| \\ &\leqslant \int \left|\mathbb{1}_{A_j^*} - \mathbb{1}_{A_j}\right| u_j\,d\bar{\mu} \\ &= \int \mathbb{1}_{A_j^* \setminus A_j \cup A_j \setminus A_j^*} u_j\,d\bar{\mu} \\ &\leqslant \int \mathbb{1}_N u_j\,d\mu = 0\end{aligned}$$

for a suitable μ-null-set $N \supset A_j^* \setminus A_j \cup A_j \setminus A_j^*$. This proves that

$$\int_{A_j^*} u_j\,d\bar{\mu} = \int_{A_j} u_j\,d\mu$$

and we see easily from this that $(u_j, \mathcal{A}_j^*)_j$ is again a (sub-, super-)martingale if $(u_j, \mathcal{A}_j)_j$ is a (sub-, super-)martingale.

Problem 17.4 To see that the condition is sufficient, set $k = j+1$. For the necessity, assume that $k = j+m$. Since $\mathcal{A}_j \subset \mathcal{A}_{j+1} \subset \cdots \subset \mathcal{A}_{j+m} = \mathcal{A}_k$ we get from the submartingale property

$$\int_A u_j\,d\mu \leqslant \int_A u_{j+1}\,d\mu \leqslant \int_A u_{j+2}\,d\mu \leqslant \cdots \leqslant \int_A u_{j+m}\,d\mu = \int_A u_k\,d\mu.$$

For supermartingales resp. martingales the conditions obviously read:

$$\int_A u_j\,d\mu \geqslant \int_A u_k\,d\mu \qquad \forall\, j<k,\ \forall\, A \in \mathcal{A}_j$$

resp.

$$\int_A u_j\,d\mu = \int_A u_k\,d\mu \qquad \forall\, j<k,\ \forall\, A \in \mathcal{A}_j.$$

Problem 17.5 We have $\mathcal{S}_j = \{A \in \mathcal{A}_j \,:\, \mu(A) < \infty\}$ and we have to check conditions (S_1)–(S_3) for a semiring, cf. page S_1. Indeed

$$\varnothing \in \mathcal{A}_j, \mu(\varnothing) = 0 \implies \varnothing \in \mathcal{S}_j \implies (S_1);$$

and

$$A, B \in \mathcal{S}_j \implies A \cap B \in \mathcal{A}_j,\ \mu(A \cap B) \leqslant \mu(A) < \infty$$

$$\implies A \cap B \in \mathcal{S}_j \implies (S_2);$$

and

$$A, B \in \mathcal{S}_j \implies A \setminus B \in \mathcal{A}_j,\ \mu(A \setminus B) \leqslant \mu(A) < \infty$$
$$\implies A \setminus B \in \mathcal{S}_j \implies (S_3).$$

Since $\mathcal{S}_j \subset \mathcal{A}_j$ also $\sigma(\mathcal{S}_j) \subset \mathcal{A}_j$. On the other hand, if $A \in \mathcal{A}_j$ with $\mu(A) = \infty$ we can, because of σ-finiteness find a sequence $(A_k)_k \subset \mathcal{A}_0 \subset \mathcal{A}_j$ such that $\mu(A_k) < \infty$ and $A_k \uparrow X$. Thus, $A_k \cap A \in \mathcal{S}_j$ for all k and $A = \bigcup_k (A_k \cap A)$. This shows that $\mathcal{A}_j \subset \sigma(\mathcal{S}_j)$.

The rest of the problem is identical to remark 17.2(i) when combined with Lemma 15.6.

Problem 17.6 Using Corollary 12.11 we can approximate $u_j \in \mathcal{L}^2(\mathcal{A}_j)$ by simple functions in $\mathcal{E}(\mathcal{A}_j)$, i.e. with functions of the form $f_j^\ell = \sum_m c_j^{\ell,m} \mathbb{1}_{A_j^{\ell,m}}$ (the sum is a finite sum!) where $c_j^\ell \in \mathbb{R}$ and $A_j^\ell \in \mathcal{A}_j$. Using the Cauchy-Schwarz inequality we also see that

$$\int (f_j^\ell - u_j) u_j \, d\mu \leqslant \|f_j^\ell - u_j\|_{L^2} \cdot \|u_j\|_{L^2} \xrightarrow[j \text{ fixed}]{\ell \to \infty} 0.$$

Using the martingale property we find for $j \leqslant k$:

$$\int \mathbb{1}_{A_j^{\ell,m}} u_k \, d\mu = \int \mathbb{1}_{A_j^{\ell,m}} u_j \, d\mu \qquad \forall\, \ell,\ m$$

and therefore

$$\int f_j^\ell u_k \, d\mu = \int f_j^\ell u_j \, d\mu \qquad \forall\, \ell$$

and since the limit $\ell \to \infty$ exists

$$\int u_j u_k \, d\mu = \lim_\ell \int f_j^\ell u_k \, d\mu = \lim_\ell \int f_j^\ell u_j \, d\mu = \int u_j^2 \, d\mu.$$

Problem 17.7 Since the f_j's are bounded, it is clear that $(f \bullet u)_k$ is integrable. Now take $A \in \mathcal{A}_k$. Then

$$\begin{aligned} \int_A (f \bullet u)_{k+1} \, d\mu &= \int_A \sum_{j=1}^{k+1} f_{j-1}(u_j - u_{j-1}) \, d\mu \\ &= \int_A (f \bullet u)_k + f_k(u_{k+1} - u_k) \, d\mu \\ &= \int_A (f \bullet u)_k \, d\mu + \int (\mathbb{1}_A \cdot f_k)(u_{k+1} - u_k) \, d\mu \end{aligned}$$

Using Remark 17.2(iii) we find

$$\begin{aligned} \int (\mathbb{1}_A \cdot f_k)(u_{k+1} - u_k) \, d\mu &= \int \mathbb{1}_A \cdot f_k u_{k+1} \, d\mu - \int \mathbb{1}_A \cdot f_k u_k \, d\mu \\ &= \int \mathbb{1}_A \cdot f_k u_k \, d\mu - \int \mathbb{1}_A \cdot f_k u_k \, d\mu \\ &= 0 \end{aligned}$$

and we conclude that

$$\int_A (f \bullet u)_{k+1} \, d\mu = \int_A (f \bullet u)_k \, d\mu \qquad \forall\, A \in \mathcal{A}_k.$$

Problem 17.8 (i) Note that

$$S_{n+1}^2 - S_n^2 = (S_n + X_{n+1})^2 - S_n^2 = X_{n+1}^2 + X_{n+1}S_n.$$

If $A \in \mathcal{A}_n$, then $\mathbb{1}_A S_n$ is independent of X_{n+1} and we find, therefore,

$$\begin{aligned}\int_A (S_{n+1}^2 - S_n^2)\,dP &= \int_A X_{n+1}^2\,dP + \int_A X_{n+1}S_n\,dP \\ &\geqslant \int_A X_{n+1}S_n\,dP \\ &= \int X_{n+1}(\mathbb{1}_A S_n)\,dP \\ &= \underbrace{\int X_{n+1}\,dP}_{=0} \int \mathbb{1}_A S_n\,dP \\ &= 0.\end{aligned}$$

(ii) Observe, first of all, that due to independence

$$\begin{aligned}\int S_n^2\,dP &= \sum_{j=1}^n \int X_j^2\,dP + \sum_{j\neq k} \int X_j X_k\,dP \\ &= n\int X_1^2\,dP + \sum_{j\neq k} \underbrace{\int X_j\,dP}_{=0} \int X_k\,dP \\ &= n\int X_1^2\,dP\end{aligned}$$

so that $\kappa := \int X_1^2\,dP$ is a reasonable candidate for the assertion. Using the calculation of part (i) of this problem we see

$$[S_{n+1}^2 - \kappa(n+1)] - [S_n^2 - \kappa n] = X_{n+1}^2 + X_{n+1}S_n - \kappa$$

and integrating over $\int_A \dots dP$ for any $A \in \mathcal{A}_n$ gives, just as in (i), because of independence of $\mathbb{1}_A$ and X_{n+1} resp. $\mathbb{1}_A S_n$ and X_{n+1}

$$\begin{aligned}&\int_A \Big([S_{n+1}^2 - \kappa(n+1)] - [S_n^2 - \kappa n]\Big)\,dP \\ &= \int \mathbb{1}_A \cdot X_{n+1}^2\,dP + \int X_{n+1}\,dP \int \mathbb{1}_A \cdot S_n\,dP - \kappa \int_A dP \\ &= P(A)\int X_{n+1}^2\,dP - \kappa\int_A dP \\ &= 0\end{aligned}$$

since X_1 and X_{n+1} are identically distributed implying that $\kappa = \int X_{n+1}^2\,dP = \int X_1^2\,dP$.

Problem 17.9 As in Problem 17.8 we find

$$M_{n+1} - M_n = X_{n+1}^2 + S_n X_{n+1} - \sigma_{n+1}^2.$$

Integrating over $A \in \mathcal{A}_n$ yields

$$\int_A (M_{n+1} - M_n)\,dP$$

$$
\begin{aligned}
&= \int_A X_{n+1}^2 \, dP + \int_A S_n X_{n+1} \, dP - \sigma_{n+1}^2 \int_A dP \\
&= P(A) \underbrace{\int_\Omega X_{n+1}^2 \, dP}_{=\sigma_{n+1}^2} + \int_A S_n \, dP \underbrace{\int_\Omega X_{n+1} \, dP}_{=0} - \sigma_{n+1}^2 P(A) \\
&= 0,
\end{aligned}
$$

where we used the independence of $\mathbb{1}_A$ and X_{n+1} resp. of $\mathbb{1}_A S_n$ and X_{n+1} and the hint given in the statement of the problem.

Problem 17.10 We find that for $A \in \mathcal{A}_n$

$$
\int_A u_{n+1} \, d\mu = \int_A (u_n + d_{n+1}) \, d\mu = \int_A u_n \, d\mu + \int_A d_{n+1} \, d\mu = \int_A u_n \, d\mu
$$

which shows that $(u_n, \mathcal{A}_n)_n$ is a martingale, hence $(u_n^2, \mathcal{A}_n)_n$ is a submartingale—cf. Example 17.3(vi).

Now

$$
\int u_n^2 \, d\mu = \sum_j \int d_j^2 \, d\mu + 2 \sum_{j<k} \int d_j d_k \, d\mu
$$

but, just as in Problem 17.6, we can approximate d_j by $\mathcal{A}_j$-measurable simple functions $(f_j^\ell)_{\ell \in \mathbb{N}}$ which shows, since $\int_A d_k \, d\mu = 0$ for any $A \in \mathcal{A}_j$ and $k > j$:

$$
\int d_j d_k \, d\mu = \lim_\ell \int f_j^\ell d_k \, d\mu = 0.
$$

Problem 17.11 For $A \in \mathcal{A}_n$ we find

$$
\begin{aligned}
&\int_A \left[\left(\frac{1-p}{p} \right)^{S_{n+1}} - \left(\frac{1-p}{p} \right)^{S_n} \right] dP \\
&\qquad = \int_A \left(\frac{1-p}{p} \right)^{S_n} \left[\left(\frac{1-p}{p} \right)^{X_{n+1}} - 1 \right] dP \\
&\qquad = \int_A \left(\frac{1-p}{p} \right)^{S_n} dP \cdot \int_\Omega \left[\left(\frac{1-p}{p} \right)^{X_{n+1}} - 1 \right] dP
\end{aligned}
$$

where we used that $\mathbb{1}_A \left(\frac{1-p}{p}\right)^{S_n}$ and $\left(\frac{1-p}{p}\right)^{X_{n+1}} - 1$ are independent, see formulae (17.6) and (17.7). But since X_{n+1} is a Bernoulli random variable we find

$$
\begin{aligned}
&\int_\Omega \left[\left(\frac{1-p}{p} \right)^{X_{n+1}} - 1 \right] dP \\
&\qquad = \left[\left(\frac{1-p}{p} \right)^1 - 1 \right] \cdot p + \left[\left(\frac{1-p}{p} \right)^{-1} - 1 \right] \cdot (1-p) \\
&\qquad = [1 - 2p] + [2p - 1] \\
&\qquad = 0.
\end{aligned}
$$

The integrability conditions for martingales are obviously satisfied.

Problem 17.12 A solution in a more general context can be found in Example 19.5 on page 206 of the textbook.

Problem 17.13 By definition, a supermartingale satisfies

$$\int_A u_j \, d\mu \geqslant \int_A u_{j+1} \, d\mu \qquad \forall j \in \mathbb{N},\ A \in \mathcal{A}_j.$$

If we take $A = X$ and if $u_k = 0$, then this becomes

$$0 = \int_X u_k \, d\mu \geqslant \int_X u_{k+1} \, d\mu \geqslant 0$$

and since, by assumption, $u_{k+1} \geqslant 0$, we conclude that $u_{k+1} = 0$.

Problem 17.14 By definition,

$$A \in \mathcal{A}_\tau \iff A \in \mathcal{A} \quad \text{and} \quad \forall\, j : A \cap \{\tau \leqslant j\} \in \mathcal{A}_j.$$

Thus,

- $\varnothing \in \mathcal{A}_\tau$ is obvious;
- if $A \in \mathcal{A}_\tau$, then

  $$A^c \cap \{\tau \leqslant j\} = \{\tau \leqslant j\} \setminus A = \underbrace{\{\tau \leqslant j\}}_{\in \mathcal{A}_j} \setminus \underbrace{(A \cap \{\tau \leqslant j\})}_{\in \mathcal{A}_j} \in \mathcal{A}_j$$

  thus $A^c \in \mathcal{A}_\tau$.
- if $A_\ell \in \mathcal{A}_\tau$, $\ell \in \mathbb{N}$, then

  $$\Big[\bigcup_\ell A_\ell\Big] \cap \{\tau \leqslant j\} = \bigcup_\ell \underbrace{\big[A_\ell \cap \{\tau \leqslant j\}\big]}_{\in \mathcal{A}_j} \in \mathcal{A}_j$$

  thus $\bigcup A_\ell \in \mathcal{A}_\tau$.

Problem 17.15 By definition, τ is a stopping time if

$$\forall\, j \in \mathbb{N} : \{\tau \leqslant j\} \in \mathcal{A}_j.$$

Thus, if τ is a stopping time, we find for $j \geqslant 1$

$$\{\tau < j\} = \{\tau \leqslant j - 1\} \in \mathcal{A}_{j-1} \subset \mathcal{A}_j$$

and, therefore,

$$\{\tau = j\} = \{\tau \leqslant j\} \setminus \{\tau < j\} \in \mathcal{A}_j.$$

Conversely, if $\{\tau = j\} \in \mathcal{A}_j$ for all j, we get

$$\{\tau \leqslant k\} = \{\tau = 0\} \cup \{\tau = 1\} \cup \cdots \cup \{\tau = k\} \in \mathcal{A}_0 \cup \cdots \cup \mathcal{A}_k \subset \mathcal{A}_k.$$

Problem 17.16 Since $\sigma \wedge \tau \leqslant \sigma$ and $\sigma \wedge \tau \leqslant \tau$, we find from Lemma 17.6 that

$$\mathcal{F}_{\sigma\wedge\tau} \subset \mathcal{F}_\sigma \cap \mathcal{F}_\tau.$$

Conversely, if $A \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ we know that

$$A \cap \{\sigma \leqslant j\} \in \mathcal{F}_j \quad \text{and} \quad A \cap \{\tau \leqslant j\} \in \mathcal{F}_j \qquad \forall\, j \in \mathbb{N}_0.$$

Thus,

$$A \cap \{\sigma \wedge \tau \leqslant j\} = A \cap \big(\{\sigma \leqslant j\} \cup \{\tau \leqslant j\}\big) \in \mathcal{F}_j$$

and we get $A \in \mathcal{F}_{\sigma\wedge\tau}$.

18 Martingale convergence theorems Solutions to Problems 18.1–18.9

Problem 18.1 We have $\tau_0 = 0$ which is clearly a stopping time and since

$$\sigma_1 := \inf\{j > 0 \,:\, u_j \leqslant a\} \wedge N \qquad (\inf \varnothing = +\infty)$$

it is clear that

$$\{\sigma_1 > \ell\} = \{u_1 > a\} \cap \dots \cap \{u_\ell > a\} \in \mathcal{A}_\ell.$$

The claim follows by induction once we have shown that σ_k and τ_k are stopping times for a generic value of k. Since the structure of their definitions are similar, we show this for σ_k only.

By induction assumption, let $\tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_k$ be stopping times. By definition,

$$\sigma_k := \inf\{j > \tau_{k-1} \,:\, u_j \leqslant a\} \wedge N \qquad (\inf \varnothing = +\infty)$$

and we find for $\ell \in \mathbb{N}$ and $\ell < N$

$$\{\sigma_k > \ell\} = \{\tau_{k-1} \leqslant \ell\} \in \mathcal{A}_\ell$$

while, by definition

$$\{\sigma_k = N\} = \varnothing \in \mathcal{A}_N.$$

Problem 18.2 Theorem 18.7 becomes for supermartingales: *Let $(u_\ell)_{\ell \in -\mathbb{N}}$ be a backwards supermartingale and assume that $\mu|_{\mathcal{A}_{-\infty}}$ is σ-finite. Then $\lim_{j\to\infty} u_{-j} = u_{-\infty} \in (-\infty, \infty]$ exists a.e. Moreover, L^1-$\lim_{j\to\infty} u_{-j} = u_{-\infty}$ if, and only if, $\sup_j \int u_{-j}\,d\mu < \infty$; in this case $(u_\ell, \mathcal{A}_\ell)_{\ell \in -\mathbb{N}}$ is a supermartingale and $u_{-\infty}$ is finitely-valued.*

Using this theorem the claim follows immediately from the supermartingale property:

$$-\infty < \int_A u_{-1}\,d\mu \leqslant \int_A u_{-j}\,d\mu \leqslant \int_A u_{-\infty}\,d\mu < \infty \quad \forall\, j \in \mathbb{N},\ A \in \mathcal{A}_{-\infty}$$

and, in particular, for $A = X \in \mathcal{A}_{-\infty}$.

Problem 18.3 Corollary 18.3 shows pointwise a.e. convergence. Using Fatou's lemma we get

$$\begin{aligned} 0 = \lim_{j\to\infty} \int u_j\,d\mu &= \liminf_{j\to\infty} \int u_j\,d\mu \\ &\geqslant \int \liminf_{j\to\infty} u_j\,d\mu \end{aligned}$$

$$= \int u_\infty \, d\mu \geqslant 0$$

so that $u_\infty = 0$ a.e.

Moreover, since $\int u_j \, d\mu \xrightarrow{j\to\infty} 0 = \int u_\infty \, d\mu$, Theorem 18.6 shows that $u_j \to u_\infty$ in L^1-sense.

Problem 18.4 From $L^1\text{-}\lim_{j\to\infty} u_j = f$ we conclude that $\sup_j \int |u_j| \, d\mu < \infty$ and we get that $\lim_{j\to\infty} u_j$ exists a.e. Since L^1-convergence also implies a.e. convergence of a subsequence, the limiting functions must be the same.

Problem 18.5 The quickest solution uses the famous Chung-Fuchs result that a simple random walk (this is just $S_j := X_1 + \cdots + X_j$ with X_k iid Bernoulli $p = q = \frac{1}{2}$) does not converge and that $-\infty = \liminf_j S_j < \limsup_j S_j = \infty$ a.e. Knowing this we are led to

$$P(u_j \text{ converges}) = P(X_0 + 1 = 0) = \frac{1}{2}.$$

It remains to show that u_j is a martingale. For $A \in \sigma(X_1, \dots, X_j)$ we get

$$\begin{aligned}
\int_A u_{j+1} \, dP &= \int_A (X_0 + 1)(X_1 + \cdots + X_j + X_{j+1}) \, dP \\
&= \int_A (X_0 + 1)(X_1 + \cdots + X_j) \, dP + \int_A (X_0 + 1) X_{j+1} \, dP \\
&= \int_A u_j \, dP + \int_A (X_0 + 1) \, dP \int_\Omega X_{j+1} \, dP \\
&= \int_A u_j \, dP
\end{aligned}$$

where the last step follows because of independence.

If you do not know the Chung-Fuchs result, you could argue as follows: assume that for some finite random variable S the limit $S_j(\omega) \to S(\omega)$ takes place on a set $A \subset \Omega$. Since the X_j's are iid, we have

$$X_2 + X_3 + \cdots \to S$$

and

$$X_1 + X_2 + \cdots \to S$$

which means that S and $S + X_1$ have the same probability distribution. But this entails that S is necessarily $\pm\infty$, i.e., S_j cannot have a finite limit.

Problem 18.6 (i) Cf. the construction in Scholium 17.4.

(ii) Note that $n^2 - (n-1)^2 - 1 = 2n - 2$ is even.

The function $f : \mathbb{R}^{2n-2} \to \mathbb{R}$, $f(x_1, \dots, x_{2n-2}) = x_1 + \cdots + x_{n^2-(n-1)^2}$ is clearly Borel measurable, i.e. the function

$$f(X_{(n-1)^2+2}, \dots, X_{n^2}) = X_{(n-1)^2+2} + \cdots + X_{n^2}$$

is $\mathcal{A}_n$-measurable and so is the set A_n. Moreover, $x \in A_n$ if, and only if, exactly half of $X_{(n-1)^2+2}, \dots, X_{n^2}$ are $+1$ and the other half is -1. Thus,

$$\lambda(A_n) = \binom{2n-2}{n-1} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right)^{n-1} = \binom{2n-2}{n-1} \left(\frac{1}{2}\right)^{2n-2}$$

Using Stirling's formula, we get

$$\begin{aligned}\frac{1}{2^{2k}}\binom{2k}{k} &= \frac{(2k)!}{k!\,k!}\\ &\sim \frac{\sqrt{2\pi 2k}(2k)^{2k}e^k e^k}{2^{2k}\sqrt{2\pi k}\sqrt{2\pi k}k^k k^k e^{2k}}\\ &= \frac{1}{\sqrt{k\pi}} \xrightarrow{k\to\infty} 0.\end{aligned}$$

Setting $k = n-1$ this shows both

$$\lim_n \lambda(A_n) = 0 \quad\text{and}\quad \sum_n \lambda(A_n) \sim \sum_n \frac{1}{\sqrt{n}} = \infty.$$

Finally, $\limsup_n \mathbb{1}_{A_n} = \mathbb{1}_{\limsup_n A_n} = 1$ a.e. while, by Fatou's lemma

$$0 \leqslant \int \liminf_n \mathbb{1}_{A_n}\,d\lambda \leqslant \liminf_n \int \mathbb{1}_{A_n}\,d\lambda = \liminf_n \lambda(A_n) = 0,$$

i.e., $\liminf_n \mathbb{1}_{A_n} = 0$ a.e. This means that $\mathbb{1}_{A_n}$ does not have a limit as $n\to\infty$.

(iii) For $A \in \mathcal{A}_n$ we have because of independence

$$\begin{aligned}&\int_A M_{n+1}\,d\lambda\\ &= \int_A M_n(1+X_{n^2+1})\,d\lambda + \int_A \mathbb{1}_{A_n}X_{n^2+1}\,d\lambda\\ &= \int_A M_n\,d\lambda \int_{[0,1]}(1+X_{n^2+1})\,d\lambda + \int_A \mathbb{1}_{A_n}\,d\lambda \int_{[0,1]} X_{n^2+1}\,d\lambda\\ &= \int_A M_n\,d\lambda.\end{aligned}$$

(iv) We have

$$\begin{aligned}&\{M_{n+1}\neq 0\}\\ &= \{M_{n+1}\neq 0, X_{n^2+1}=-1\} \cup \{M_{n+1}\neq 0, X_{n^2+1}=+1\}\\ &\subset A_n \cup \{M_n \neq 0, X_{n^2+1}=+1\}.\end{aligned}$$

(v) By definition,

$$M_{n+1}-M_n = M_nX_{n^2+1} + \mathbb{1}_{A_n}X_{n^2+1} = (M_n+\mathbb{1}_{A_n})X_{n^2+1}$$

so that

$$|M_{n+1}-M_n| = |M_n+\mathbb{1}_{A_n}|\cdot|X_{n^2+1}| = |M_n+\mathbb{1}_{A_n}|.$$

This shows that for $x\in\{\lim_n M_n \text{ exists}\}$ the limit $\lim_n \mathbb{1}_{A_n}(x)$ exists. But, because of (ii), the latter is a null set, so that the pointwise limit of M_n cannot exist.

On the other hand, using the inequality (iv), shows

$$\lambda(M_{n+1}\neq 0) \leqslant \tfrac{1}{2}\lambda(M_n\neq 0) + \lambda(A_n)$$

and iterating this gives

$$\lambda(M_{n+k}\neq 0) \leqslant \tfrac{1}{2^k}\lambda(M_n\neq 0) + \lambda(A_n) + \cdots\lambda(A_{n+k-1})$$

$$\leqslant \tfrac{1}{2^k} + \lambda(A_n) + \cdots \lambda(A_{n+k-1}).$$

Letting first $n \to \infty$ and then $k \to \infty$ yields

$$\limsup_j \lambda(M_j \neq 0) = 0$$

so that $\lim_j \lambda(M_j = 0) = 0$.

Problem 18.7 Note that for $A \in \big\{\{1\},\{2\},\dots,\{n\},\{n+1,n+2,\dots\}\big\}$ we have

$$\begin{aligned}\int_A X_{n+1}\,dP &= \int_A (n+2)\mathbb{1}_{[n+2,\infty)\cap\mathbb{N}}\,dP\\ &= \begin{cases} 0 & \text{if } A \text{ is a singleton}\\ \int_{[n+1,\infty)\cap\mathbb{N}} (n+2)\mathbb{1}_{[n+2,\infty)\cap\mathbb{N}}\,dP & \text{else}\end{cases}\end{aligned}$$

and in the second case we have

$$\begin{aligned}\int_{[n+1,\infty)\cap\mathbb{N}} (n+2)\mathbb{1}_{[n+2,\infty)\cap\mathbb{N}}\,dP &= \int_{[n+2,\infty)\cap\mathbb{N}} (n+2)\,dP\\ &= (n+2)\sum_{j=n+2}^{\infty} P(\{j\})\\ &= (n+2)\sum_{j=n+2}^{\infty}\left(\frac{1}{j} - \frac{1}{j+1}\right)\\ &= 1,\end{aligned}$$

The same calculation shows

$$\begin{aligned}\int_A X_n\,dP &= \int_A (n+1)\mathbb{1}_{[n+1,\infty)\cap\mathbb{N}}\,dP\\ &= \begin{cases} 0 & \text{if } A \text{ is a singleton}\\ \int_{[n+1,\infty)\cap\mathbb{N}} (n+1)\mathbb{1}_{[n+1,\infty)\cap\mathbb{N}}\,dP = 1 & \text{else}\end{cases}\end{aligned}$$

so that

$$\int_A X_{n+1}\,dP = \int_A X_n\,dP$$

for all A from a generator of the σ-algebra which contains an exhausting sequence. This shows, by Remark 17.2(i) that $(X_n)_n$ is indeed a martingale.

The second calculation from above also shows that $\int X_n\,dP = 1$ while

$$\sup_n X_n = \infty \qquad \text{and} \qquad \lim_n X_n = 0$$

are obvious.

Problem 18.8 (i) Using Problem 17.6 we get

$$\begin{aligned}\int (u_j - u_{j-1})^2\,d\mu &= \int u_j^2\,d\mu - 2\int u_j u_{j-1}\,d\mu + \int u_{j-1}^2\,d\mu\\ &= \int u_j^2\,d\mu - 2\int u_{j-1}^2\,d\mu + \int u_{j-1}^2\,d\mu\end{aligned}$$

$$= \int u_j^2 \, d\mu - \int u_{j-1}^2 \, d\mu$$

which means that

$$\int u_N^2 \, d\mu = \sum_{j=1}^N \int (u_j - u_{j-1})^2 \, d\mu$$

and the claim follows.

(ii) Because of Example 17.3(vi), $p = 2$, we conclude that $(u_j^2)_j$ is a submartingale which, due to L^2-boundedness, satisfies the assumptions of Theorem 18.2 on submartingale convergence. This means that $\lim_j u_j^2 = u^2$ exists a.e. This is, alas, not good enough to get $u_j \to u$ a.e., it only shows that $|u_j| \to |u|$ a.e.

The following trick helps: let $(A_k)_k \subset \mathcal{A}_0$ be an exhausting sequence with $A_k \uparrow X$ and $\mu(A_k) < \infty$. Then $(\mathbb{1}_{A_k} u_j)_j$ is an L^1-bounded martingale: indeed, if $A \in \mathcal{A}_n$ then $A \cap A_k \in \mathcal{A}_n$ and it is clear that

$$\int_A \mathbb{1}_{A_k} u_n \, d\mu = \int_{A \cap A_k} u_n \, d\mu = \int_{A \cap A_k} u_{n+1} \, d\mu = \int_A \mathbb{1}_{A_k} u_{n+1} \, d\mu$$

while, by the Cauchy-Schwarz inequality,

$$\int |\mathbb{1}_{A_k} u_n| \, d\mu \leqslant \sqrt{\mu(A_k)} \cdot \sqrt{\sup_n \int u_n^2 \, d\mu} \leqslant c_k.$$

Thus, we can use Theorem 18.2 and conclude that

$$\mathbb{1}_{A_k} u_n \xrightarrow{n \to \infty} \mathbb{1}_{A_k} u$$

almost everywhere with, because of almost-everywhere-uniqueness of the limits on each of the sets A_k, a single function u. This shows $u_n \to u$ a.e.

(iii) Following the hint and using the arguments of part (i) we find

$$\begin{aligned}
\int (u_{j+k} - u_j)^2 \, d\mu &= \int (u_{j+k}^2 - u_j^2) \, d\mu \\
&= \sum_{\ell=j+1}^{j+k} \int (u_\ell^2 - u_{\ell-1}^2) \, d\mu \\
&= \sum_{\ell=j+1}^{j+k} \int (u_\ell - u_{\ell-1})^2 \, d\mu.
\end{aligned}$$

Now we use Fatou's lemma and the result of part (ii) to get

$$\begin{aligned}
\int \liminf_j (u - u_j)^2 \, d\mu &\leqslant \liminf_j \int (u - u_j)^2 \, d\mu \\
&\leqslant \limsup_j \int (u - u_j)^2 \, d\mu \\
&\leqslant \limsup_j \sum_{\ell=j+1}^{\infty} \int (u_\ell - u_{\ell-1})^2 \, d\mu \\
&= 0
\end{aligned}$$

since, by L^2-boundedness, $\sum_{k=1}^\infty \int (u_k - u_{k-1})^2 \, d\mu < \infty$.

(iv) Since $\mu(X) < \infty$, constants are integrable and we find using the Cauchy-Schwarz and Markov inequalities

$$\begin{aligned}\int_{|u_k|>R} |u_k|\,d\mu &\leqslant \sqrt{\mu(|u_k|>R)}\cdot\sqrt{\int u_k^2\,d\mu}\\ &\leqslant \frac{1}{R}\sqrt{\int u_k^2\,d\mu}\cdot\sqrt{\int u_k^2\,d\mu}\\ &\leqslant \frac{1}{R}\sup_k \int u_k^2\,d\mu\end{aligned}$$

from which we get uniform integrability; the claim follows now from parts (i)–(iii) and Theorem 18.6.

Problem 18.9 (i) Note that $\int \epsilon_j\,dP = 0$ and $\int \epsilon_j^2\,dP = 1$. Moreover, $X_n := \sum_{j=1}^n \epsilon_j y_j$ is a martingale w.r.t. the filtration $\mathcal{A}_n := \sigma(\epsilon_1,\dots,\epsilon_n)$ and

$$\int X_n^2\,dP = \sum_{j=1}^n y_j^2.$$

Problem 18.8 now shows that $\sum_{j=1}^\infty y_j^2 < \infty$ means that the martingale $(X_n)_n$ is L^2-bounded, i.e. X_n converges a.e. The converse follows from part (iii).

(ii) This follows with the same arguments as in part (i) with $\mathcal{A}_n = \sigma(Y_1,\dots,Y_n)$.

(iii) We show that $S_n^2 - A_n$ is a martingale. Now for $A \in \mathcal{A}_n$

$$\begin{aligned}\int_A M_{n+1}\,dP &= \int_A (S_{n+1}^2 - A_{n+1})\,dP\\ &= \int_A (S_n^2 + 2X_{n+1}S_n + X_{n+1}^2 - A_n - \sigma_{n+1}^2)\,dP\\ &= \int_A (S_n^2 - A_n)\,dP + \int_A (2X_{n+1}S_n + X_{n+1}^2 - \sigma_{n+1}^2)\,dP\\ &= \int_A M_n\,dP + \int_A (2X_{n+1}S_n + X_{n+1}^2 - \sigma_{n+1}^2)\,dP\end{aligned}$$

But, because of independence,

$$\begin{aligned}&\int_A (2X_{n+1}S_n + X_{n+1}^2 - \sigma_{n+1}^2)\,dP\\ &= \int_A 2X_{n+1}\,dP \int_\Omega S_n\,dP + P(A)\int X_{n+1}^2\,dP - P(A)\sigma_{n+1}^2\\ &= 0 + P(A)\sigma_{n+1}^2 - P(A)\sigma_{n+1}^2\\ &= 0.\end{aligned}$$

and the claim is established.

Now define

$$\tau := \tau_\kappa := \inf\{j\,:\,|M_j| > \kappa\}.$$

By optional sampling, $(M_{n\wedge\tau_\kappa})_n$ is again a martingale and we have

$$\begin{aligned}|M_{n\wedge\tau}| &= M_n \mathbb{1}_{\{n<\tau\}} + |M_\tau|\mathbb{1}_{\{n\geqslant\tau\}}\\ &\leqslant \kappa\mathbb{1}_{\{n<\tau\}} + |M_\tau|\mathbb{1}_{\{n\geqslant\tau\}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \kappa \mathbb{1}_{\{n<\tau\}} + |M_\tau - M_{\tau-1}| \mathbb{1}_{\{n\geqslant\tau\}} + |M_{\tau-1}| \mathbb{1}_{\{n\geqslant\tau\}} \\
&= \kappa \mathbb{1}_{\{n<\tau\}} + |X_\tau| \mathbb{1}_{\{n\geqslant\tau\}} + |M_{\tau-1}| \mathbb{1}_{\{n\geqslant\tau\}} \\
&\leqslant \kappa \mathbb{1}_{\{n<\tau\}} + |X_\tau| \mathbb{1}_{\{n\geqslant\tau\}} + \kappa \mathbb{1}_{\{n\geqslant\tau\}} \\
&\leqslant \kappa + C
\end{aligned}
$$

where we used, for the estimate of $M_{\tau-1}$, the definition of τ for the last estimate. Since $(M_{n\wedge\tau})_n$ is a martingale, this gives

$$\int (S_{n\wedge\tau}^2 - A_{n\wedge\tau})\, dP = \int (S_0^2 - A_0)\, dP = 0$$

so that

$$\int A_{n\wedge\tau}\, dP = \int S_{n\wedge\tau}^2\, dP \leqslant (\kappa + C)^2$$

uniformly in n.

Thus, by Beppo Levi's theorem,

$$\int A_\tau\, dP \leqslant (\kappa + C)^2 < \infty$$

which means that $A_\tau < \infty$ almost surely. But since $\sum_j X_j$ converges almost surely, $P(\tau = \infty) = 1$ for sufficiently large κ, and we are done.

19 The Radon-Nikodým Theorem and other applications of martingales Solutions to Problems 19.1–19.18

Problem 19.1 This problem is intimately linked with problem 19.7.

Without loss of generality we assume that μ and ν are finite measures, the case for σ-finite μ and arbitrary ν is exactly as in the proof of Theorem 19.2.

Let $(A_j)_j$ be as described in the statement of the problem and define the finite σ-algebras $\mathcal{A}_n := \sigma(A_1, \dots, A_n)$. Using the hint we can achieve that

$$\mathcal{A}_n = \sigma\big(C_1^n, \dots, C_{\ell(n)}^n\big)$$

with mutually disjoint C_j^k's and $\ell(n) \leqslant 2^n + 1$ and $\dot\bigcup_j C_j^n = X$. Then the construction of Example 19.5 yields a countably-indexed martingale since the σ-algebras $\mathcal{A}_j$ are increasing.

This means, that the countable version of the martingale convergence theorem is indeed enough for the proof.

Problem 19.2 Using simply the Radon-Nikodým theorem, Theorem 19.2, gives

$$\forall\, t \quad \exists\, p_t(x) \;\text{ such that }\; \nu_t(dx) = p_t(x) \cdot \mu_t(dx)$$

with a measurable function $x \mapsto p_t(x)$; it is, however, far from being clear that $(t, x) \mapsto p_t(x)$ is jointly measurable.

A slight variation of the proof of Theorem 19.2 allows us to incorporate parameters provided the families of measures are measurable w.r.t. these parameters. Following the hint we set (notation as in the proof of 19.2)

$$p_\alpha(t, x) := \sum_{A \in \alpha} \frac{\nu_t(A)}{\mu_t(A)} I_A(x)$$

with the agreement that $\frac{0}{0} := 0$ (note that $\frac{a}{0}$ with $a \neq 0$ will not turn up because of the absolute continuity of the measures!). Since $t \mapsto \mu_t(A)$ and $t \mapsto \mu_t(A)$ are measurable, the above sum is measurable so that

$$(t, x) \mapsto p(t, x)$$

is a jointly measurable function. If we can show that

$$\lim_\alpha p_\alpha(t, x) = p(t, x)$$

exists (say, in L^1, t being fixed) then the limiting function is again jointly measurable.

Using exactly the arguments of the proof of Theorem 19.2 with t fixed we can confirm that this limit exists and defines a jointly measurable function with the property that

$$\nu_t(dx) = p(t,x) \cdot \nu_t(dx).$$

Because of the a.e. uniqueness of the Radon-Nikodým density the functions $p(t,x)$ and $p_t(x)$ coincide, for every t a.e. as functions of x; without additional assumptions on the nature of the dependence on the parameter, the exceptional set may, though, depend on t!

Problem 19.3 We write $u^\pm$ for the positive resp. negative parts of $u \in \mathcal{L}^1(\mathcal{A})$, i.e. $u = u^+ - u^-$ and $u^\pm \geqslant 0$. Fix such a function u and define

$$\nu^\pm(F) := \int_F u^\pm(x)\,\mu(dx), \quad \forall\, F \in \mathcal{F}.$$

Clearly, $\nu^\pm$ are measures on the σ-algebra $\mathcal{F}$. Moreover

$$\forall\, N \in \mathcal{F},\ \mu(N) = 0 \implies \nu^\pm(N) = \int_N u^\pm\,d\mu = 0$$

which means that $\nu^\pm \ll \mu$. By the Radon-Nikodým theorem, Theorem 19.2 and its Corollary 19.6, we find (up to null-sets unique) positive functions $f^\pm \in \mathcal{L}^1(\mathcal{F})$ such that

$$\nu^\pm(F) = \int_F f^\pm\,d\mu \quad \forall\, F \in \mathcal{F}.$$

Thus, $u^\mathcal{F} := f^+ - f^- \in \mathcal{L}^1(\mathcal{F})$ clearly satisfies

$$\int_F u^\mathcal{F}\,d\mu = \int_F u\,d\mu \quad \forall\, F \in \mathcal{F}.$$

To see uniqueness, we assume that $w \in \mathcal{L}^1(\mathcal{F})$ also satisfies

$$\int_F w\,d\mu = \int_F u\,d\mu \quad \forall\, F \in \mathcal{F}.$$

Since then

$$\int_F u^\mathcal{F}\,d\mu = \int_F w\,d\mu \quad \forall\, F \in \mathcal{F}.$$

we can choose $f := \{w > u^\mathcal{F}\}$ and find

$$0 = \int_{\{w > u^\mathcal{F}\}} (w - u^\mathcal{F})\,d\mu$$

which is only possible if $\mu(\{w > u^\mathcal{F}\}) = 0$. Similarly we conclude that $\mu(\{w < u^\mathcal{F}\}) = 0$ from which we get $w = u^\mathcal{F}$ almost everywhere.

Reformulation of the submartingale property.

Recall that $(u_j, \mathcal{A}_j)_j$ is a submartingale if, for every j, $u_j \in \mathcal{L}^1(\mathcal{A}_j)$ and if

$$\int_A u_j\,d\mu \leqslant \int_A u_{j+1}\,d\mu \quad \forall A \in \mathcal{A}_j,\ \forall\, j.$$

We claim that this is equivalent to saying

$$u_j \leqslant u_{j+1}^{\mathcal{A}_j} \quad \text{almost everywhere, } \forall\, j.$$

The direction '⇒' is clear. To see '⇐' we fix j and observe that, since

$$\int_A u_j\,d\mu \leqslant \int_A u_{j+1}\,d\mu = \int_A u_{j+1}^{\mathcal{A}_j}\,d\mu \quad \forall A \in \mathcal{A}_j,$$

we get, in particular, for $A := \{u_{j+1}^{\mathcal{A}_j} < u_j\} \in \mathcal{A}_j$,

$$0 \leqslant \int_{\{u_{j+1}^{\mathcal{A}_j} < u_j\}} (u_{j+1}^{\mathcal{A}_j} - u_j)\,d\mu$$

which is only possible if $\mu(\{u_{j+1}^{\mathcal{A}_j} < u_j\}) = 0$.

Problem 19.4 The assumption $\nu \leqslant \mu$ immediately implies $\nu \ll \mu$. Indeed,

$$\mu(N) = 0 \implies 0 \leqslant \nu(N) \leqslant \mu(N) = 0 \implies \nu(N) = 0.$$

Using the Radon-Nikodým theorem, Theorem 19.2 we conclude that there exists a measurable function $f \in \mathcal{M}^+(\mathcal{A})$ such that $\nu = f \cdot \mu$. Assume that $f > 1$ on a set of positive μ-measure. Without loss of generality we may assume that the set has finite measure, otherwise we would consider the intersection $A_k \cap \{f > 1\}$ with some exhausting sequence $A_k \uparrow X$ and $\mu(A_k) < \infty$.

Then, for sufficiently small $\epsilon > 0$ we know that $\mu(\{f \geqslant 1 + \epsilon\}) > 0$ and so

$$\begin{aligned}
\nu(\{f \geqslant 1 + \epsilon\}) &= \int_{\{f \geqslant 1+\epsilon\}} f\,d\mu \\
&\geqslant (1 + \epsilon) \int_{\{f \geqslant 1+\epsilon\}} d\mu \\
&\geqslant (1 + \epsilon)\mu(\{f \geqslant 1 + \epsilon\}) \\
&\geqslant \mu(\{f \geqslant 1 + \epsilon\})
\end{aligned}$$

which is impossible.

Problem 19.5 Because of our assumption both $\mu \ll \nu$ and $\nu \ll \mu$ which means that we know

$$\nu = f\mu \quad \text{and} \quad \mu = g\nu$$

for positive measurable functions f, g which are a.e. unique. Moreover,

$$\nu = f\mu = f \cdot g\nu$$

so that $f \cdot g$ is almost everywhere equal to 1 and the claim follows.

Because of Corollary 19.6 it is clear that $f, g < \infty$ a.e. and, by the same argument, $f, g > 0$ a.e.

Note that we do not have to specify *w.r.t. which measure* we understand the 'a.e.' since their null sets coincide anyway.

Problem 19.6 Take Lebesgue measure $\lambda := \lambda^1$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and the function $f(x) := x + \infty \cdot \mathbb{1}_{[0,1]^c}(x)$. Then $f \cdot \lambda$ is certainly not σ-finite.

Problem 19.7 Since both μ and ν are σ-finite, we can restrict ourselves, using the technique of the Proof of Theorem 19.2 to the case where μ and ν are finite. All we have to do is to pick an exhaustion $(K_\ell)_\ell$, $K_\ell \uparrow X$ such that $\mu(K_\ell), \mu(K_\ell) < \infty$ and to consider the measures $\mathbb{1}_{K_\ell}\mu$ and $\mathbb{1}_{K_\ell}\nu$ which clearly inherit the absolute continuity from μ and ν.

Using the Radon-Nikodým theorem (Theorem 19.2) we get that

$$\mu_j \ll \nu_j \implies \mu_j = u_j \cdot \nu_j$$

with an $\mathcal{A}_j$-measurable positive density u_j. Moreover, since μ is a finite measure,

$$\int_X u_j \, d\nu = \int_X u_j \, d\nu_j = \int_X d\mu_j = \mu_j(X) < \infty$$

so that all the $(u_j)_j$ are ν-integrable. Using exactly the same argument as at the beginning of the proof of Theorem 19.2 (ii)⇒(i), we get that $(u_j)_j$ is even uniformly ν-integrable. Finally, $(u_j)_j$ is a martingale (given the measure ν), since for $j, j+1$ and $A \in \mathcal{A}_j$ we have

$$\begin{aligned}
\int_A u_{j+1} \, d\nu &= \int_A u_{j+1} \, d\nu_{j+1} \\
&= \int_A d\mu_{j+1} && (u_{j+1} \cdot \nu_{j+1} = \mu_{j+1}) \\
&= \int_A d\mu_j && (A \in \mathcal{A}_j) \\
&= \int_A u_j \, d\nu_j && (\mu_j = u_j \cdot \nu_j) \\
&= \int_A u_j \, d\nu
\end{aligned}$$

and we conclude that $u_j \to u_\infty$ a.e. and in $L^1(\nu)$ for some limiting function u_∞ which is still $L^1(\nu)$ and also $\mathcal{A}_\infty := \sigma(\bigcup_{j \in \mathbb{N}} \mathcal{A}_j)$-measurable. Since, by assumption, $\mathcal{A}_\infty = \mathcal{A}$, this argument shows also that

$$\mu = u_\infty \cdot \nu$$

and it reveals that

$$u_\infty = \frac{d\mu}{d\nu} = \lim_j \frac{d\mu_j}{d\nu_j}.$$

Problem 19.8 This problem is somewhat ill-posed. We should first embed it into a suitable context, say, on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Denote by $\lambda = \lambda^1$ one-dimensional Lebesgue measure. Then

$$\mu = \mathbb{1}_{[0,2]}\lambda \quad \text{and} \quad \nu = \mathbb{1}_{[1,3]}\lambda$$

and from this it is clear that

$$\nu = \mathbb{1}_{[1,2]}\nu + \mathbb{1}_{(2,3]}\nu = \mathbb{1}_{[1,2]}\lambda + \mathbb{1}_{(2,3]}\lambda$$

and from this we read off that

$$\mathbb{1}_{[1,2]}\nu \ll \mu$$

while

$$\mathbb{1}_{(2,3]}\nu \perp \mu.$$

It is interesting to note how 'big' the null-set of ambiguity for the Lebesgue decomposition is—it is actually $\mathbb{R} \setminus [0,3]$ a, from a Lebesgue (i.e. λ) point of view, huge and infinite set, but from a μ-ν-perspective a negligible, name null, set.

Problem 19.9 Since we deal with a bounded measure we can use $F(x) := \mu(-\infty, x)$ rather than the more cumbersome definition for F employed in Problem 7.9 (which is good for locally finite measures!).

With respect to one-dimensional Lebesgue measure λ we can decompose μ according to Theorem 19.9 into

$$\mu = \mu^\circ + \mu^\perp \quad \text{where} \quad \mu^\circ \ll \lambda,\ \ \mu^\perp \perp \lambda.$$

Now *define* $\mu_2 := \mu^\circ$ and $F_2 := \mu^\circ(-\infty, x)$. We have to prove property (2). For this we observe that μ° is a finite measure (since $\mu^\circ \leqslant \mu$ and that, therefore, $\mu^\circ = f \cdot \lambda$ with a function $f \in L^1(\lambda)$. Thus, for every $R > 0$

$$\begin{aligned} F(y_j) - F(x_j) &= \mu^\circ(x_j, y_j) \\ &= \int_{(x_j,y_j)} f(t)\,\lambda(dt) \\ &\leqslant R\int_{(x_j,y_j)} \lambda(dt) + \lambda\big(\{f \geqslant R\} \cap (x_j, y_j)\big) \\ &\leqslant R\int_{(x_j,y_j)} \lambda(dt) + \frac{1}{R}\int_{(x_j,y_j)} f\,d\lambda \end{aligned}$$

where we used the Markov inequality, cf. Proposition 10.12, in the last step. Summing over $j = 1, 2, \ldots, N$ gives

$$\sum_{j=1}^{N} |F_2(y_j) - F_2(x_j)| \leqslant R \cdot \delta + \frac{1}{R}\int f\,d\lambda$$

since $\dot\bigcup_j (x_j, y_j) \subset \mathbb{R}$. Now we choose for given $\epsilon > 0$

$$\text{first} \quad R := \frac{\int f\,d\lambda}{\epsilon} \qquad \text{and then} \quad \delta := \frac{\epsilon}{R} = \frac{\epsilon^2}{\int f\,d\lambda}$$

to confirm that

$$\sum_{j=1}^{N} |F_2(y_j) - F_2(x_j)| \leqslant 2\epsilon$$

this settles (2).

Now consider the measure $\mu^\perp$. Its distribution function $F^\perp(x) := \mu^\perp(-\infty, x)$ is increasing, left-continuous but not necessarily continuous. Such a function has, by Lemma 13.12 at most countably many discontinuities (jumps), which we denote by J. Thus, we can write

$$\mu^\perp = \mu_1 + \mu_3$$

with the jump (or saltus) $\Delta F(y) := F(y+) - F(y-)$ if $y \in J$.

$$\mu_1 := \sum_{y \in J} \Delta F(y) \cdot \delta_y, \quad \text{and} \quad \mu_3 := \mu^\perp - \mu_1;$$

μ_1 is clearly a measure (the sum being countable) with $\mu_1 \leqslant \mu^\perp$ and so is, therefore, μ_2 (since the defining difference is always positive). The corresponding distribution functions are

$$F_1(x) := \sum_{y \in J, y < x} \Delta F(y)$$

(called the jump or saltus function) and

$$F_2(x) := F^\perp(x) - F_1(x).$$

It is clear that F_2 is increasing and, more importantly, continuous so that the problem is solved.

It is interesting to note that our problem shows that we can decompose every left- or right-continuous monotone function into an absolutely continuous and singular part and the singular part again into a continuous and discontinuous part:

$$g = g_{\text{ac}} + g_{\text{sc}} + g_{\text{sd}}$$

where

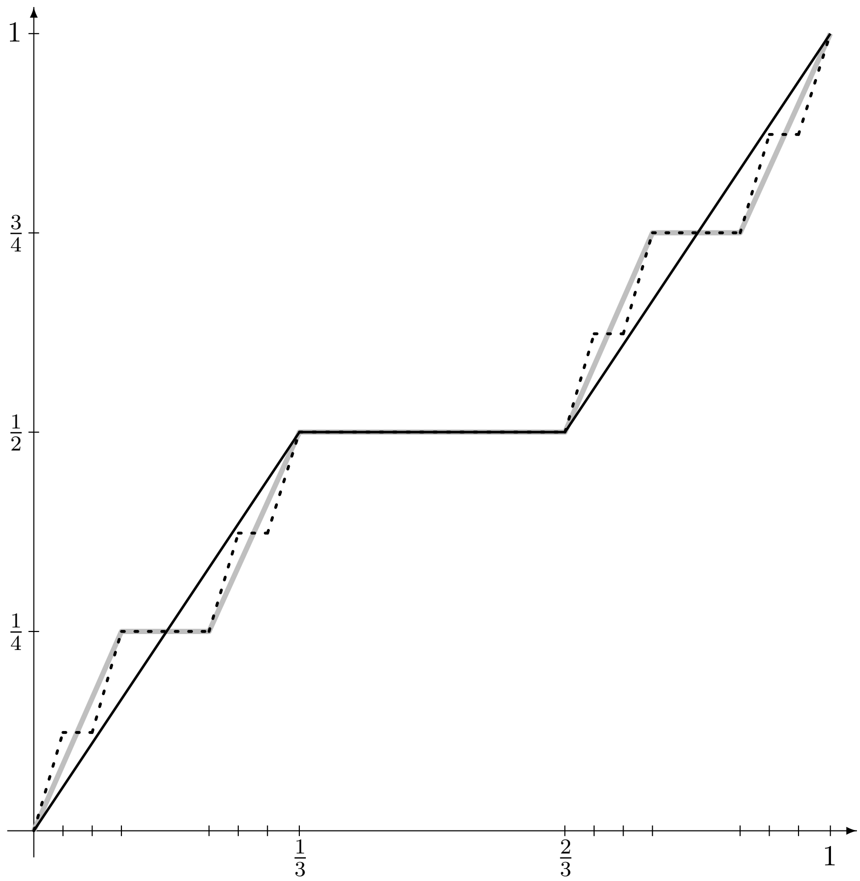
g —is a monotone left- or right-continuous function;

g_{ac} —is a monotone absolutely continuous (and in particular continuous) function;

g_{sc} —is a monotone continuous but singular function;

g_{sd} —is a monotone discontinuous (even: pure jump), but nevertheless left- or right-continuous, and singular function.

Problem 19.10 (i) In the following picture F_1 is represented by a black line, F_2 by a grey line and F_3 is a dotted black line.



(ii),(iii) The construction of the F_k's also shows that

$$|F_k(x) - F_{k+1}(x)| \leqslant \frac{1}{2^{k+1}}$$

since we modify F_k only on a set J_{k+1}^{ℓ} by replacing a diagonal line by a combination of diagonal-flat-diagonal and all this happens only within a range of 2^{-k} units. Since the flat bit is in the middle, we get that the maximal deviation between F_k and F_{k+1} is at most $\frac{1}{2} \cdot 2^{-k}$. Just look at the pictures!

Thus the convergence of $F_k \to F$ is uniform, i.e. it preserves continuity and F is continuous as all the F_k's are. That F is increasing is already inherited from the pointwise limit of the F_k's:

$$\begin{aligned} x < y &\implies \forall\, k\ :\ F_k(x) \leqslant F_k(y) \\ &\implies F(x) = \lim_k F_k(x) \leqslant \lim_k F_k(y) = F(y). \end{aligned}$$

(iv) Let C denote the Cantor set. Then for $x \in [0,1] \setminus C$ we find k and ℓ such that $x \in J_k^{\ell}$ (which is an open set!) and, since on those pieces F_k and F do not differ any more

$$F_k(x) = F(x) \implies F'(x) = F_k'(x) = 0$$

where we used that $F_k|J_k^{\ell}$ is constant. Since $\lambda(C) = 0$ (see Problem 7.10) we have $\lambda([0,1] \setminus C) = 1$ so that F' exists a.e. and satisfies $F' = 0$ a.e.

(v) We have $J_k^\ell = (a_\ell, b_\ell)$ (we suppress the dependence of a_ℓ, b_ℓ on k with, because of our ordering of the middle-thirds sets (see the problem):

$$a_1 < b_1 < a_2 < \cdots < a_{2^k-1} < b_{2^k-1}$$

and

$$\sum_{\ell=1}^{2^k-1} \big[F(b_\ell) - F(a_\ell)\big] = F(b_{2^k-1}) - F(a_1) \xrightarrow{k\to\infty} F(1) - F(0) = 1$$

while (with the convention that $a_0 := 0$)

$$\sum_{\ell=1}^{2^k-1} (a_\ell - b_{\ell-1}) \xrightarrow{k\to\infty} 0.$$

This leads to a contradiction since, because of the first equality, the sum

$$\sum_{\ell=1}^{2^k-1} \big[F(a_\ell) - F(b_{\ell-1})\big]$$

will never become small.

Problem 19.11 We can assume that $VX_j < \infty$, otherwise the inequality would be trivial.

Note that the random variables $X_j - EX_j$, $j = 1, 2, \ldots, n$ are still independent and, of course, centered (= mean-zero). Thus, by Example 17.3(x) we get that

$$M_k := \sum_{j=1}^k (X_j - EX_j) \quad \text{is a martingale}$$

and, because of Example 17.3(v), $(|M_k|)_k$ is a submartingale. Applying (19.12) in this situation proves the claimed inequality since

$$\begin{aligned} VM_n &= E(M_n^2) && \text{(since } EM_n = 0\text{)} \\ &= \sum_{j=1}^n E(X_j^2) \end{aligned}$$

where we used, for the last equality, what probabilists call *Theorem of Bienaymé* for the independent random variables X_j:

$$\begin{aligned} E(M_n^2) &= \sum_{j,k=1}^n E\big[(X_j - EX_j)(X_k - EX_k)\big] \\ &= \sum_{j=k=1}^n E\big[(X_j - EX_j)^2\big] + \sum_{j\neq k} E\big[(X_j - EX_j)\big]E\big[(X_k - EX_k)\big] \quad \text{(by independence)} \\ &= \sum_{j=k=1}^n E\big[(X_j - EX_j)^2\big] \\ &= \sum_{j=1}^n E\big[M_j^2\big] \\ &= \sum_{j=1}^n VM_j. \end{aligned}$$

Problem 19.12 (i) As in the proof of Theorem 19.12 we find

$$\begin{aligned}\int u^p\,d\mu &\overset{(13.8)}{=} p\int_0^\infty s^{p-1}\,\mu\left(\{u\geqslant s\}\right)\,ds\\ &\leqslant p\int_0^\infty s^{p-2}\bigg(\int \mathbb{1}_{\{u\geqslant s\}}(x)w(x)\mu(dx)\bigg)\,ds\\ &= p\int\bigg(\int_0^\infty \mathbb{1}_{[0,u(x)]}(s)s^{p-2}\,ds\bigg)w(x)\,\mu(dx)\\ &= p\int\frac{u(x)^{p-1}}{p-1}\,w(x)\,\mu(dx)\\ &= \frac{p}{p-1}\int u^{p-1}w\,d\mu\end{aligned}$$

Note that this inequality is meant in $[0,+\infty]$, i.e. we allow the cases $a\leqslant+\infty$ and $+\infty\leqslant+\infty$.

(ii) Pick conjugate numbers $p,q\in(1,\infty)$, i.e. $q=\frac{p}{p-1}$. Then we can rewrite the result of (i) and then apply Hölder's inequality to get

$$\begin{aligned}\|u\|_p^p &\leqslant \frac{p}{p-1}\int u^{p-1}w\,d\mu\\ &\leqslant \frac{p}{p-1}\bigg(\int u^{(p-1)q}\,d\mu\bigg)^{1/q}\bigg(\int w^p\,d\mu\bigg)^{1/p}\\ &= \frac{p}{p-1}\bigg(\int u^p\,d\mu\bigg)^{1-1/p}\|w\|_p\\ &= \frac{p}{p-1}\|u\|_p^{p-1}\cdot\|w\|_p\end{aligned}$$

and the claim follows upon dividing both sides by $\|u\|_p^{p-1}$. (Here we use the finiteness of this expression, i.e. the assumption $u\in\mathcal{L}^p$).

Problem 19.13 Only the first inequality needs proof. Note that

$$\max_{1\leqslant j\leqslant N}\int|u_j|^p\,d\mu\leqslant\int\max_{1\leqslant j\leqslant N}|u_j|^p\,d\mu=\int u_N^*\,d\mu$$

from which the claim easily follows.

Problem 19.14 Let $(A_k)_k\subset\mathcal{A}_0$ be an exhausting sequence, i.e. $A_k\uparrow X$ and $\mu(A_k)<\infty$. Since $(u_j)_j$ is L^1-bounded, we know that

$$\sup_j\|u_j\|_p\leqslant c<\infty$$

and we find, using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$

$$\int|\mathbb{1}_{A_k}u_j|\,d\mu\leqslant\big(\mu(A_k)\big)^{1/q}\cdot\|u_j\|_p\leqslant c\big(\mu(A_k)\big)^{1/q}$$

uniformly for all $j\in\mathbb{N}$. This means that the martingale $(\mathbb{1}_{A_k}u_j)_j$ (see the solution to Problem 18.8) is L^1-bounded and we get, as in Problem 18.8 that for some unique function u

$$\lim_j\mathbb{1}_{A_k}u_j=\mathbb{1}_{A_k}u\quad\forall\,k$$

a.e., hence $u_j \xrightarrow{j\to\infty} u$ a.e. Using Fatou's Lemma we get

$$\begin{aligned}\int |u|^p\,d\mu &= \int \liminf_j |u_j|^p\,d\mu \\ &\leqslant \liminf \int |u_j|^p\,d\mu \\ &\leqslant \sup_j \int |u_j|^p\,d\mu \;<\; \infty\end{aligned}$$

which means that $u \in L^p$.

For each $k \in \mathbb{N}$ the martingale $(\mathbb{1}_{A_k} u_j)_j$ is also uniformly integrable: using Hölder's and Markov's inequalities we arrive at

$$\begin{aligned}\int_{\{\mathbb{1}_{A_k}|u_j| > \mathbb{1}_{A_k} R\}} \mathbb{1}_{A_k}|u_j|\,d\mu &\leqslant \int_{\{|u_j|>R\}} \mathbb{1}_{A_k}|u_j|\,d\mu \\ &\leqslant \big(\mu\{|u_j| > R\}\big)^{1/q}\|u_j\|_p \\ &\leqslant \left(\frac{1}{R^p}\,\|u_j\|_p^p\right)^{1/q}\|u_j\|_p \\ &\leqslant \frac{c^{p/q+1}}{R^{p/q}}\end{aligned}$$

and the latter tends, uniformly for all j, to zero as $R \to \infty$. Since $\mathbb{1}_{A_k} \cdot R$ is integrable, the claim follows.

Thus, Theorem 18.6 applies and shows that for $u_\infty := u$ and every $k \in \mathbb{N}$ the family $(u_j \mathbb{1}_{A_k})_{j \in \mathbb{N}\cup\{\infty\}}$ is a martingale. Because of Example 17.3(vi) $(|u_j|^p \mathbb{1}_{A_k})_{j\in\mathbb{N}\cup\{\infty\}}$ is a submartingale and, therefore, for all $k \in \mathbb{N}$

$$\int |\mathbb{1}_{A_k} u_j|^p\,d\mu \leqslant \int |\mathbb{1}_{A_k} u_{j+1}|^p\,d\mu \leqslant \int |\mathbb{1}_{A_k} u_\infty|^p\,d\mu = \int |\mathbb{1}_{A_k} u|^p\,d\mu,$$

Since, by Fatou's lemma

$$\int |\mathbb{1}_{A_k} u|^p\,d\mu = \int \liminf_j |\mathbb{1}_{A_k} u_j|^p\,d\mu \leqslant \liminf_j \int |\mathbb{1}_{A_k} u_j|^p\,d\mu$$

we see that

$$\int |\mathbb{1}_{A_k} u|^p\,d\mu = \lim_j \int |\mathbb{1}_{A_k} u_j|^p\,d\mu = \sup_j \int |\mathbb{1}_{A_k} u_j|^p\,d\mu.$$

Since suprema interchange, we get

$$\begin{aligned}\int |u|^p\,d\mu &= \sup_k \int |\mathbb{1}_{A_k} u|^p\,d\mu \\ &= \sup_k \sup_j \int |\mathbb{1}_{A_k} u_j|^p\,d\mu \\ &= \sup_j \sup_k \int |\mathbb{1}_{A_k} u_j|^p\,d\mu \\ &= \sup_j \int |u_j|^p\,d\mu\end{aligned}$$

and Riesz's convergence theorem, Theorem 12.10, finally proves that $u_j \to u$ in L^p.

Problem 19.15 Since f_k is a martingale and since

$$\begin{aligned}\int |f_k|\,d\mu &\leqslant \sum_{z\in 2^{-k}\mathbb{Z}Z^n} \frac{1}{\lambda^n(Q_k(z))} \int_{Q_k(z)} |f|\,d\lambda^n \int \mathbb{1}_{Q_k(z)}\,d\lambda^n \\ &= \sum_{z\in 2^{-k}\mathbb{Z}Z^n} \int_{Q_k(z)} |f|\,d\lambda^n \\ &= \int |f|\,d\lambda^n < \infty\end{aligned}$$

we get from the martingale convergence theorem 18.2 that

$$f_\infty := \lim_k f_k$$

exists almost everywhere and that $f_\infty \in \mathcal{L}^1(\mathcal{B})$. The above calculation shows, on top of that, that for any set $Q \in \mathcal{A}_k^{[0]}$

$$\int_Q f_k\,d\lambda^n = \int_Q f\,d\lambda^n$$

and

$$\int_Q |f_k|\,d\lambda^n \leqslant \int_Q |f|\,d\lambda^n$$

which means that, using Fatou's Lemma,

$$\int_Q |f_\infty|\,d\lambda^n \leqslant \liminf_k \int_Q |f_k|\,d\lambda_n \leqslant \int_Q |f|\,d\lambda_n$$

for all $Q \in \mathcal{A}_k^{[0]}$ and any k. Since $\mathcal{S} = \bigcup_k \mathcal{A}_k^{[0]}$ is a semi-ring and since on both sides of the above inequality we have measures, this inequality extends to $\mathcal{B} = \sigma(\mathcal{S})$ (cf. Lemma 15.6) and we get

$$\int_B |f_\infty|\,d\lambda^n \leqslant \int_B |f|\,d\lambda^n.$$

Since f_∞ and f are $\mathcal{B}$-measurable, we can take $B = \{|f_\infty| > |f|\}$ and we get that $f = f_\infty$ almost everywhere. This shows that $(f_k)_{k\in\mathbb{N}\cup\{\infty\}}$ is a martingale.

Thus all conditions of Theorem 18.6 are satisfied and we conclude that $(f_k)_k$ is uniformly integrable.

Problem 19.16 As one would expect, the derivative at x turns out to be $u(x)$. This is seen as follows (without loss of generality we can assume that $y > x$):

$$\begin{aligned}&\left|\frac{1}{x-y}\left(\int_{[a,x]} u(t)\,dt - \int_{[a,x]} u(t)\,dt\right) - u(x)\right| \\ &= \left|\frac{1}{x-y}\int_{[x,y]} \big(u(t)-u(x)\big)\,dt\right| \\ &\leqslant \frac{1}{|x-y|}\int_{[x,y]} \big|u(t)-u(x)\big|\,dt \\ &\leqslant \frac{1}{|x-y|}\,|x-y| \sup_{t\in[x,y]} \big|u(t)-u(x)\big| \\ &= \sup_{t\in[x,y]} \big|u(t)-u(x)\big|\end{aligned}$$

and the last expression tends to 0 as $|x-y| \to 0$ since u is uniformly continuous on compact sets.

If u is not continuous but merely of class L^1, we have to refer to Lebesgue's differentiation theorem, Theorem 19.20, in particular formula (19.21) which reads in our case

$$u(x) = \lim_{r\to 0} \frac{1}{2r} \int_{(x-r,x+r)} u(t)\,dt$$

for Lebesgue almost every $x \in (a,b)$.

Problem 19.17 We follow the hint: first we remark that by Lemma 13.12 we know that f has at most countably many discontinuities. Since it is monotone, we also know that $F(t) := f(t+) = \lim_{s>t, s\to t} f(s)$ exists and is finite for every t and that $\{f \neq F\}$ is at most countable (since it is contained in the set of discontinuities of f), hence a Lebesgue null set.

If f is right-continuous, $\mu(a,b] := f(b) - f(a)$ extends uniquely to a measure on the Borel-sets and this measure is locally finite and σ-finite. If we apply Theorem 19.9 to μ and $\lambda = \lambda^1$ we can write $\mu = \mu^\circ + \mu^\perp$ with $\mu^\circ \ll \lambda$ and $\mu^\perp \perp \lambda$. By Corollary 19.22 $D\mu^\perp = 0$ a.e. and $D\mu^\circ$ exists a.e. and we get a.e.

$$D\mu(x) = \lim_{r\to 0} \frac{\mu(x-r,x+r)}{2r} = \lim_{r\to 0} \frac{\mu^\circ(x-r,x+r)}{2r} + 0$$

and we can set $f'(x) = D\mu(x)$ which is a.e. defined. Where it is not defined, we put it equal to 0.

Now we get

$$\begin{aligned} f(b) - f(a) &= \mu(a,b] \\ &\geqslant \mu(a,b) \\ &= \int_{(a,b)} d\mu \\ &\geqslant \int_{(a,b)} d\mu^\circ \\ &= \int_{(a,b)} D\mu(x)\,\lambda(dx) \\ &= \int_{(a,b)} f'(x)\,\lambda(dx). \end{aligned}$$

The above estimates show that we get equality if f is continuous and also absolutely continuous w.r.t. Lebesgue measure.

Problem 19.18 Without loss of generality we may assume that $f_j(a) = 0$, otherwise we would consider the (still increasing) functions $x \mapsto f_j(x) - f_j(a)$ resp. their sum $x \mapsto s(x) - s(a)$. The derivatives are not influenced by this operation. As indicated in the hint call $s_n(x) := f_1(x) + \cdots + f_n(x)$ the nth partial sum. Clearly, s, s_n are increasing

$$\frac{s_n(x+h) - s_n(x)}{h} \leqslant \frac{s_{n+1}(x+h) - s_{n+1}(x)}{h} \leqslant \frac{s(x+h) - s(x)}{h}.$$

and possess, because of Problem 19.17, almost everywhere positive derivatives:

$$s'_n(x) \leqslant s'_{n+1}(x) \leqslant \cdots s'(x), \qquad \forall\, x \notin E$$

Note that the exceptional null-sets depend originally on the function s_n etc. but we can consider their (countable!!) union and get thus a universal exceptional null set E. This shows that the formally differentiated series

$$\sum_{j=1}^{\infty} f'_j(x) \quad \text{converges for all } x \notin E.$$

Since the sequence of partial sums is increasing, it will be enough to check that

$$s'(x) - s'_{n_k}(x) \xrightarrow{k\to\infty} 0 \quad \forall x \notin E.$$

Since, by assumption the sequence $s_k(x) \to s(x)$ we can choose a subsequence n_k in such a way that

$$s(b) - s_{n_k}(b) < 2^{-k} \quad \forall\, k \in \mathbb{N}.$$

Since

$$0 \leqslant s(x) - s_{n_k}(x) \leqslant s(b) - s_{n_k}(b)$$

the series

$$\sum_{k=1}^{\infty}(s(x) - s_{n_k}(x)) \leqslant \sum_{k=1}^{\infty} 2^{-k} < \infty \quad \forall\, x \in [a,b].$$

By the first part of the present proof, we can differentiate this series term-by-term and get that

$$\sum_{k=1}^{\infty}(s'(x) - s'_{n_k}(x)) \;\; \text{converges} \quad \forall\, x \in (a,b) \setminus E$$

and, in particular, $s'(x) - s'_{n_k}(x) \xrightarrow{k\to\infty} 0$ for all $x \in (a,b) \setminus E$ which was to be proved.

20 Inner Product Spaces
Solutions to Problems 20.1–20.6

Problem 20.1 If we set $\mu = \delta_1 + \cdots + \delta_n$, $X = \{1, 2, \ldots, n\}$, $\mathcal{A} = \mathcal{P}(X)$ or $\mu = \sum_{j\in\mathbb{N}} \delta_j$, $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(X)$, respectively, we can deduce 20.5(i) and (ii) from 20.5(iii).

Let us, therefore, only verify (iii). Without loss of generality (see Scholium 20.1 and also the complexification of a real inner product space in Problem 20.3) we can consider the real case where $L^2 = L^2_{\mathbb{R}}$.

- L^2 is a vector space — this was done in Remark 12.5.
- $\langle u, v\rangle$ is finite on $L^2 \times L^2$ — this is the Cauchy-Schwarz inequality 12.3.
- $\langle u, v\rangle$ is bilinear — this is due to the linearity of the integral.
- $\langle u, v\rangle$ is symmetric — this is obvious.
- $\langle v, v\rangle$ is definite, and $\|u\|_2$ is a Norm — cf. Remark 12.5.

Problem 20.2 (i) We prove it for the complex case—the real case is simpler. Observe that

$$\begin{aligned}\langle u \pm w, u \pm w\rangle &= \langle u, u\rangle \pm \langle u, w\rangle \pm \langle w, u\rangle + \langle w, w\rangle \\ &= \langle u, u\rangle \pm \langle u, w\rangle \pm \overline{\langle u, w\rangle} + \langle w, w\rangle \\ &= \langle u, u\rangle \pm 2\,\mathrm{Re}\langle u, w\rangle + \langle w, w\rangle.\end{aligned}$$

Thus,

$$\langle u + w, u + w\rangle + \langle u - w, u - w\rangle = 2\langle u, u\rangle + 2\langle w, w\rangle.$$

Since $\|v\|^2 = \langle v, v\rangle$ we are done.

(ii) (SP_1): Obviously,

$$0 < (u, u) = \tfrac{1}{4}\,\|2v\|^2 = \|v\|^2 \implies v \neq 0.$$

(SP_1): is clear.

(iii) Using at the point (*) below the parallelogram identity, we have

$$\begin{aligned}4(u + v, w) &= 2(u + v, 2w) \\ &= \tfrac{1}{2}\left(\|u + v + 2w\|^2 - \|u + v - 2w\|^2\right) \\ &= \tfrac{1}{2}\left(\|(u + w) + (v + w)\|^2 - \|(u - w) + (v - w)\|^2\right) \\ &\overset{*}{=} \tfrac{1}{2}\Big[2\big(\|u + w\|^2 + \|v + w\|^2 - \|u - w\|^2 - \|v - w\|^2\big)\Big]\end{aligned}$$

$$= 4(u,w) + 4(v,w)$$

and the claim follows.

(iv) We show $(qv,w) = q(v,w)$ for all $q \in \mathbb{Q}$. If $q = n \in \mathbb{N}_0$, we iterate (iii) n times and have

$$(nv,w) = n(v,w) \qquad \forall\, n \in \mathbb{N}_0 \tag{*}$$

(the case $n = 0$ is obvious). By the same argument, we get for $m \in \mathbb{N}$

$$(v,w) = \big(m\,\tfrac{1}{m}\,v,w\big) = m\big(\tfrac{1}{m}\,v,w\big)$$

which means that

$$\big(\tfrac{1}{m}\,v,w\big) = \tfrac{1}{m}\,(v,w) \qquad \forall\, m \in \mathbb{N}. \tag{**}$$

Combining (*) and (**) then yields $(\frac{n}{m}\,v,w) = \frac{n}{m}\,(v,w)$. Thus,

$$(pu + qv, w) = p(u,w) + q(v,w) \qquad \forall\, p,q \in \mathbb{Q}.$$

(v) By the lower triangle inequality for norms we get for any $s,t \in \mathbb{R}$

$$\begin{aligned}\big|\|tv \pm w\| - \|sv \pm w\|\big| &\leqslant \|(tv \pm w) - (sv \pm w)\|\\ &= \|(t-s)v\|\\ &= |t-s|\cdot\|v\|.\end{aligned}$$

This means that the maps $t \mapsto tv \pm w$ are continuous and so is $t \mapsto (tv,w)$ as the sum of two continuous maps. If $t \in \mathbb{R}$ is arbitrary, we pick a sequence $(q_j)_{j\in\mathbb{N}} \subset \mathbb{Q}$ such that $\lim_j q_j = t$. Then

$$(tv,w) = \lim_j (q_j v, w) = \lim_j q_j (qv, w) = t(v,w)$$

so that

$$(su + tv, w) = (su,w) + (tv,w) = s(u,w) + t(v,w).$$

Problem 20.3 This is actually a problem on complexification of inner product spaces... .

Since v and iw are vectors in $V \oplus iV$ and since $\|v\| = \|\pm iv\|$, we get

$$\begin{aligned}(v,iw)_{\mathbb{R}} &= \tfrac{1}{4}\left(\|v+iw\|^2 - \|v-iw\|^2\right)\\ &= \tfrac{1}{4}\left(\|i(w-iv)\|^2 - \|(-i)(w+iv)\|^2\right)\\ &= \tfrac{1}{4}\left(\|w-iv\|^2 - \|w+iv\|^2\right) \qquad (*)\\ &= (w,-iv)_{\mathbb{R}}\\ &= -(w,iv)_{\mathbb{R}}.\end{aligned}$$

In particular,

$$(v, iv) = -(v, iv) \implies (v, iv) = 0 \quad \forall v,$$

and we get

$$(v, v)_{\mathbb{C}} = (v, v)_{\mathbb{R}} > 0 \implies v = 0.$$

Moreover, using (*) we see that

$$\begin{aligned}(v, w)_{\mathbb{C}} &= (v, w)_{\mathbb{R}} + i(v, iw)_{\mathbb{R}} \\ &\overset{*}{=} (w, v)_{\mathbb{R}} - i(w, iv)_{\mathbb{R}} \\ &= (w, v)_{\mathbb{R}} + \bar{i} \cdot (w, iv)_{\mathbb{R}} \\ &= \overline{(w, v)_{\mathbb{R}} + i(w, iv)_{\mathbb{R}}} \\ &= \overline{(w, v)_{\mathbb{C}}}.\end{aligned}$$

Finally, for real $\alpha, \beta \in \mathbb{R}$ the linearity property of the real scalar product shows that

$$\begin{aligned}(\alpha u + \beta v, w)_{\mathbb{C}} &= \alpha(u, w)_{\mathbb{R}} + \beta(v, w)_{\mathbb{R}} + i\alpha(u, iw)_{\mathbb{R}} + i\beta(v, iw)_{\mathbb{R}} \\ &= \alpha(u, w)_{\mathbb{C}} + \beta(v, w)_{\mathbb{C}}.\end{aligned}$$

Therefore to get the general case where $\alpha, \beta \in \mathbb{C}$ we only have to consider the purely imaginary case:

$$\begin{aligned}(iv, w)_{\mathbb{C}} = (iv, w)_{\mathbb{R}} + i(iv, iw)_{\mathbb{R}} &\overset{*}{=} -(v, iw)_{\mathbb{R}} - i(v, -w)_{\mathbb{R}} \\ &= -(v, iw)_{\mathbb{R}} + i(v, w)_{\mathbb{R}} \\ &= i\big(i(v, iw)_{\mathbb{R}} + (v, w)_{\mathbb{R}}\big) \\ &= i(v, w)_{\mathbb{C}},\end{aligned}$$

where we used twice the identity (*). This shows complex linearity in the first coordinate, while skew-linearity follows from the conjugation rule $(v, w)_{\mathbb{C}} = \overline{(w, v)_{\mathbb{C}}}$.

Problem 20.4 The parallelogram law (stated for L^1) *would* say:

$$\left(\int_0^1 |u + w|\, dx\right)^2 + \left(\int_0^1 |u - w|\, dx\right)^2 = 2\left(\int_0^1 |u|\, dx\right)^2 + 2\left(\int_0^1 |w|\, dx\right)^2.$$

If $u \pm w, u, w$ have always only ONE sign (i.e. +ve or −ve), we could leave the modulus signs $|\bullet|$ away, and the equality would be correct! To show that there is no equality, we should therefore choose functions where we have some sign change. We try:

$$u(x) = 1/2, \qquad w(x) = x$$

(note: $u - w$ does change its sign!) and get

$$\begin{aligned}\int_0^1 |u + w|\, dx &= \int_0^1 (\tfrac{1}{2} + x)\, dx = [\tfrac{1}{2}(x + x^2)]_0^1 = 1 \\ \int_0^1 |u - w|\, dx &= \int_0^{1/2} (\tfrac{1}{2} - x)\, dx + \int_{1/2}^1 (x - \tfrac{1}{2})\, dx\end{aligned}$$

$$
\begin{aligned}
&= \left[\tfrac{1}{2}(x - x^2)\right]_0^{1/2} + \left[\tfrac{1}{2}(x^2 - x)\right]_{1/2}^1 \\
&= \tfrac{1}{4} - \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{1}{4} = \tfrac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 |u|\,dx &= \int_0^1 \tfrac{1}{2}\,dx = \tfrac{1}{2} \\
\int_0^1 |w|\,dx &= \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}
\end{aligned}
$$

This shows that

$$1^2 + (\tfrac{1}{4})^2 = \tfrac{17}{16} \neq 1 = 2(\tfrac{1}{2})^2 + 2(\tfrac{1}{2})^2.$$

We conclude, in particular, that L^1 cannot be a Hilbert space (since in any Hilbert space the Parallelogram law is true....).

Problem 20.5 (i) If $k = 0$ we have $\theta = 1$ and everything is obvious. If $k \neq 0$, we use the summation formula for the geometric progression to get

$$S := \frac{1}{n}\sum_{j=1}^n \theta^{jk} = \frac{1}{n}\sum_{j=1}^n \left(\theta^k\right)^j = \frac{\theta}{n}\,\frac{1-(\theta^k)^n}{1-\theta^k}$$

but $(\theta^k)^n = \exp(2\pi\frac{i}{n}\cdot k\cdot n) = \exp(2\pi i k) = 1$. Thus $S = 0$ and the claim follows.

(ii) Note that $\overline{\theta^j} = \theta^{-j}$ so that

$$
\begin{aligned}
\|v + \theta^j w\|^2 &= \langle v + \theta^j w, v + \theta^j w\rangle \\
&= \langle v, v\rangle + \langle v, \theta^j w\rangle + \langle \theta^j w, v\rangle + \langle \theta^j w, \theta^j w\rangle \\
&= \langle v, v\rangle + \theta^{-j}\langle v, w\rangle + \theta^j\langle w, v\rangle + \theta^j\theta^{-j}\langle w, w\rangle \\
&= \langle v, v\rangle + \theta^{-j}\langle v, w\rangle + \theta^j\langle w, v\rangle + \langle w, w\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{n}\sum_{j=1}^n \theta^j\|v + \theta^j w\|^2 \\
&= \frac{1}{n}\sum_{j=1}^n \theta^j\langle v, v\rangle + \frac{1}{n}\sum_{j=1}^n \langle v, w\rangle + \frac{1}{n}\sum_{j=1}^n \theta^{2j}\langle w, v\rangle + \frac{1}{n}\sum_{j=1}^n \theta^j\langle w, w\rangle \\
&= 0 + \langle v, w\rangle + 0 + 0
\end{aligned}
$$

where we used the result from part (i) of the exercise.

(iii) Since the function $\phi \mapsto e^{i\phi}\|v + e^{i\phi}w\|^2$ is bounded and continuous, the integral exists as a (proper) Riemann integral, and we can use *any* Riemann sum to approximate the integral, see 11.6–11.10 in Chapter 11 or Corollary E.6 and Theorem E.8 of Appendix E. Before we do that, we change variables according to $\psi = (\phi + \pi)/2\pi$ so that $d\psi = d\phi/2\pi$ and

$$\frac{1}{2\pi}\int_{(-\pi,\pi]} e^{i\phi}\left\|v + e^{i\phi}w\right\|^2 d\phi = -\int_{(0,1]} e^{2\pi i\psi}\left\|v - e^{2\pi i\psi}w\right\|^2 d\psi.$$

Now using equidistant Riemann sums with step $1/n$ and nodes $\theta_n^j = e^{2\pi i\cdot\frac{1}{n}\cdot j}$, $j = 1, 2, \dots, n$ yields, because of part (ii) of the problem,

$$
\begin{aligned}
-\int_{(0,1]} e^{2\pi i\psi} \left\| v - e^{2\pi i\psi} w \right\|^2 d\psi &= -\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \theta_n^j \| v - \theta_n^j w \|^2 \\
&= -\lim_{n\to\infty} \langle v, -w \rangle \\
&= \langle v, w \rangle.
\end{aligned}
$$

Problem 20.6 We assume that V is a $\mathbb{C}$-inner product space. Then,

$$
\begin{aligned}
\|v + w\|^2 &= \langle v + w, v + w \rangle \\
&= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\
&= \|v\|^2 + \langle v, w \rangle + \overline{\langle v, w \rangle} + \|w\|^2 \\
&= \|v\|^2 + 2\,\mathrm{Re}\langle v, w \rangle + \|w\|^2.
\end{aligned}
$$

Thus

$$
\|v + w\|^2 = \|v\|^2 + \|w\|^2 \iff \mathrm{Re}\langle v, w \rangle = 0 \iff v \perp w.
$$

21 Hilbert space $\mathfrak{H}$ Solutions to Problems 21.1–21.13

Problem 21.1 Let $(h_k)_k \subset \mathfrak{H}$ such that $\lim_k \|h_k - h\| = 0$. By the triangle inequality

$$\|h_k - h_\ell\| \leqslant \underbrace{\|h_k - h\|}_{\to 0} + \underbrace{\|h - h_\ell\|}_{\to 0} \xrightarrow{k,\ell\to\infty} 0.$$

Problem 21.2 Let $g, \tilde{g} \in \mathfrak{H}$. By the Cauchy-Schwarz inequality 20.3

$$|\langle g, h\rangle - \langle \tilde{g}, h\rangle| \leqslant |\langle g - \tilde{g}, h\rangle| \leqslant \|h\| \cdot \|\tilde{g} - g\|$$

which proves continuity. Incidentally, this calculation shows also that, since $g \mapsto \langle g, h\rangle$ is linear, it would have been enough to check continuity at the point $g = 0$ (think about it!).

Problem 21.3 Definiteness (N_1) and positive homogeneity (N_2) are obvious. The triangle inequality reads in this context ($g, g', h, h' \in \mathfrak{H}$):

$$|||(g,h) + (g',h')||| \leqslant |||(g,h)||| + |||(g',h')||| \iff$$
$$\big(\|g + g'\|^p + \|h + h'\|^p\big)^{1/p} \leqslant \big(\|g\|^p + \|h\|^p\big)^{1/p} + \big(\|g'\|^p + \|h'\|^p\big)^{1/p}.$$

Since

$$\big(\|g + g'\|^p + \|h + h'\|^p\big)^{1/p} \leqslant \big(\big[\|g\|\|g'\|\big]^p + \big[\|h\| + \|h'\|\big]^p\big)^{1/p}$$

we can use the Minkowski inequality for sequences resp. in $\mathbb{R}^2$—which reads for numbers $a, A, b, B \geqslant 0$

$$\big((a+b)^p + (A+B)^p\big)^{1/p} \leqslant \big(a^p + A^p\big)^{1/p} + \big(b^p + B^p\big)^{1/p}$$

—and the claim follows.

Since $\mathbb{R}^2$ is only with the Euclidean norm a Hilbert space—the parallelogram identity fails for the norms $(|x|^p + |y|^p)^{1/p}$—this shows that also in the case at hand only $p = 2$ will be a Hilbert space norm.

Problem 21.4 For the scalar product we have for all $g, g', h, h' \in \mathfrak{H}$ such that $\|g - g'\|^2 + \|h - h'\|^2 < 1$

$$\big|\langle g - g', h - h'\rangle\big| \leqslant \|g - g'\| \cdot \|h - h'\| \leqslant \big[\|g - g'\|^2 + \|h - h'\|^2\big]^{1/2}$$

where we used the elementary inequality

$$ab \leqslant \frac{1}{2}(a^2 + b^2) \leqslant a^2 + b^2 \underbrace{\leqslant \sqrt{a^2 + b^2}}_{\text{if } a^2+b^2\leqslant 1}.$$

Since $(g,h) \mapsto \left[\|g\|^2 + \|h\|^2\right]^{1/2}$ is a norm on $\mathfrak{H} \times \mathfrak{H}$ we are done.

Essentially the same calculation applies to $(t,h) \mapsto t \cdot h$.

Problem 21.5 Assume that $\mathfrak{H}$ has a countable maximal ONS, say $(e_j)_j$. Then, by definition, every vector $h \in \mathfrak{H}$ can be approximated by a sequence made up of finite linear combinations of the $(e_j)_j$:

$$h_k := \sum_{j=1}^{n(k)} \alpha_j \cdot e_j$$

(note that $\alpha_j = 0$ is perfectly possible!). In view of problem 21.4 we can even assume that the α_j are rational numbers. This shows that the set

$$\mathcal{D} := \Big\{ \sum_{j=1}^{n} \alpha_j \cdot e_j \;:\; \alpha_j \in \mathbb{Q}, n \in \mathbb{N} \Big\}$$

is a countable dense subset of $\mathfrak{H}$.

Conversely, if $\mathcal{D} \subset \mathfrak{H}$ is a countable dense subset, we can use the Gram-Schmidt procedure and obtain from $\mathcal{D}$ an ONS. Then Theorem 21.15 proves the claim.

Problem 21.6 Let us, first of all, show that for a closed subspace $C \subset \mathfrak{H}$ we have $C = (C^\perp)^\perp$.

Because of Lemma 21.4 we know that $C \subset (C^\perp)^\perp$ and that $C^\perp$ is itself a closed linear subspace of $\mathfrak{H}$. Thus,

$$C \oplus C^\perp = \mathfrak{H} = C^\perp \oplus (C^\perp)^\perp.$$

Thus C cannot be a proper subspace of $(C^\perp)^\perp$ and therefore $C = (C^\perp)^\perp$.

Applying this to the obviously closed subspace $C := \mathbb{K} \cdot w = \operatorname{span}(w)$ we conclude that $\operatorname{span}(w) = \operatorname{span}(w)^{\perp\perp}$.

By assumption, $M_w = \{w\}^\perp$ and $M_w^\perp = \{w\}^{\perp\perp}$ and we have $w \in \{w\}^{\perp\perp}$. The last expression is a (closed) subspace, so

$$w \in \{w\}^{\perp\perp} \implies \operatorname{span}(w) \subset \{w\}^{\perp\perp}$$

also. Further

$$\begin{aligned}\{w\} \subset \operatorname{span}(w) &\implies \{w\}^\perp \supset \operatorname{span}(w)^\perp \\ &\implies \{w\}^{\perp\perp} \subset \operatorname{span}(w)^{\perp\perp} = \operatorname{span}(w)\end{aligned}$$

and we conclude that

$$\{w\}^{\perp\perp} = \operatorname{span}(w)$$

which is either $\{0\}$ or a one-dimensional subspace.

Problem 21.7 (i) By Pythagoras' Theorem 21.11

$$\|e_j - e_k\|^2 = \|e_j\|^2 + \|e_k\|^2 = 2 \qquad \forall\, j \neq k.$$

This shows that no subsequence $(e_j)_{j\in\not\,}$ can ever be a Cauchy sequence, i.e. it cannot converge.

If $h \in \mathfrak{H}$ we get from Bessel's inequality 21.11 that the series

$$\sum_j |\langle e_j, h\rangle|^2 \leqslant \|h\|^2$$

is finite, i.e. converges. Thus the sequence with elements $\langle e_j, h\rangle$ must converge to 0 as $j \to \infty$.

(ii) Parseval's equality 21.11 shows that

$$\|h\|^2 = \sum_{j=1}^{\infty} |\langle e_j, h\rangle|^2 = \sum_{j=1}^{\infty} |c_j|^2 \leqslant \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

uniformly for all $h \in Q$, i.e. Q is a bounded set.

Let $(h_\ell)_\ell \subset Q$ be a sequence with $\lim_\ell h_\ell = h$ and write $c_j := \langle e_j, h\rangle$ and $c_j^\ell := \langle e_j, h_\ell\rangle$. Because of the continuity of the scalar product

$$|c_j| = |\langle e_j, h\rangle| = \lim_\ell |\langle e_j, h_\ell\rangle| = \lim_\ell |c_j^\ell| \leqslant \frac{1}{j}$$

which means that $h \in Q$ and that Q is closed.

Let $(h_\ell)_\ell \subset Q$ be a sequence and set $c_j(\ell) := \langle e_j, h_\ell\rangle$. Using the Bolzano-Weierstraß theorem for bounded sequences we get

$$|c_1(\ell)| \leqslant 1 \implies \exists \big(c_1(\ell_j^1)\big)_j \subset \big(c_1(\ell)\big)_\ell \;:\; \lim_j c_1(\ell_j^1) = \gamma_1$$

and

$$|c_2(\ell_j^1)| \leqslant \frac{1}{2} \implies \exists \big(c_2(\ell_j^2)\big)_j \subset \big(c_2(\ell_j^1)\big)_j \;:\; \lim_j c_2(\ell_j^2) = \gamma_2$$

and, recursively,

$$|c_k(\ell_j^{k-1})| \leqslant \frac{1}{k} \implies \exists \big(c_k(\ell_j^k)\big)_j \subset \big(c_k(\ell_j^{k-1})\big)_j \;:\; \lim_j c_k(\ell_j^k) = \gamma_k$$

and since we have considered sub-sub-etc.-sequences we get

$$c_k(\ell_m^m) \xrightarrow{m\to\infty} \gamma_k \quad \forall\, k \in \mathbb{N}.$$

Thus, we have constructed a subsequence $(h_{\ell_m^m})_m \subset (h_\ell)_\ell$ with

$$\langle e_k, h_{\ell_m^m}\rangle \xrightarrow{m\to\infty} \gamma_k \quad \forall\, k \in \mathbb{N} \tag{*}$$

so that $\gamma_j \leqslant 1/j$. Setting $h = \sum_j \gamma_j e_j$ we see (by Parseval's relation) that $h \in Q$. Further,

$$\begin{aligned}\|h - h_{\ell_m^m}\|^2 &= \sum_{j=1}^{\infty} |\gamma_j - c_j(\ell_m^m)|^2 \\ &\leqslant \sum_{j=1}^{N} |\gamma_j - c_j(\ell_m^m)|^2 + \sum_{j=N+1}^{\infty} \frac{4}{j^2}.\end{aligned}$$

Letting first $m \to \infty$ we get, because of (*)

$$\sum_{j=1}^{N} |\gamma_j - c_j(\ell_m^m)|^2 \xrightarrow{m\to\infty} 0,$$

and letting $N \to \infty$ gives

$$\limsup_m \|h - h_{\ell_m^m}\|^2 \leqslant \sum_{j=N+1}^{\infty} \frac{4}{j^2} \xrightarrow{N\to\infty} 0$$

so that $\lim_m \|h - h_{\ell_m^m}\|^2 = 0$.

(iii) R cannot be compact since $(e_j)_j \subset R$ does not have any convergent subsequence, see part (i).

R is bounded since $r \in R$ if, and only if, there is some $j \in \mathbb{N}$ such that

$$\|r - e_j\| \leqslant \frac{1}{j} \leqslant 1.$$

Thus, every $r \in R$ is bounded by

$$\|r\| \leqslant \|r - e_j\| + \|e_j\| \leqslant 2.$$

R is closed. Indeed, if $x_j \in B_{1/j}(e_j)$ we see that for $j \neq k$

$$\begin{aligned}\|x_j - x_k\| &= \|(x_j - e_j) + (e_j - e_k) + (e_k - x_k)\| \\ &\geqslant \|e_j - e_k\| - \|x_j - e_j\| - \|x_k - e_k\| \\ &\overset{\text{(i)}}{\geqslant} \sqrt{2} - \frac{1}{j} - \frac{1}{k}.\end{aligned}$$

This means that any sequence $(r_j)_r \subset R$ with $\lim_j r_j = r$ is in at most finitely many of the sets $\overline{B_{1/j}(e_j)}$. But a finite union of closed sets is closed so that $r \in R$.

(iv) Assume that $\sum_j \delta_j^2 < \infty$. Then closedness, boundedness and compactness follows exactly as in part (ii) of the problem with δ_j replacing $1/j$.

Conversely, assume that S is compact. Then the sequence

$$h_\ell = \sum_{j=1}^{\ell} \delta_j e_j \in S$$

and, by compactness, there is a convergent subsequence

$$h_{\ell_k} = \sum_{j=1}^{\ell_k} \delta_j e_j \xrightarrow{k\to\infty} h.$$

By Parseval's identity we get:

$$\|h_{\ell_k}\|^2 = \sum_{j=1}^{\ell_k} \delta_j^2 \xrightarrow{k\to\infty} \sum_{j=1}^{\infty} \delta_j^2 = \|h\|^2 < \infty.$$

Problem 21.8 (i) Note that for all $g \neq 0$

$$|\langle g,h\rangle| \leqslant \|g\|\cdot\|h\| \implies \frac{|\langle g,h\rangle|}{\|g\|} \leqslant \|h\|$$

so that

$$\sup_{g\neq 0}\frac{|\langle g,h\rangle|}{\|g\|} \leqslant \|h\|.$$

Since for $g = h$ the supremum is attained, we get equality.

Further, since $\left\|\frac{g}{\|g\|}\right\| = 1$, we have

$$\sup_{g\neq 0}\frac{|\langle g,h\rangle|}{\|g\|} = \sup_{g\neq 0}\left|\left\langle \frac{g}{\|g\|}, h\right\rangle\right| = \sup_{\gamma,\ \|\gamma\|=1}|\langle \gamma,h\rangle|.$$

Finally,

$$\|h\| = \sup_{g,\ \|g\|=1}|\langle g,h\rangle| \leqslant \sup_{g,\ \|g\|\leqslant 1}|\langle g,h\rangle| \leqslant \sup_{g,\ \|g\|\leqslant 1}\|g\|\cdot\|h\| \leqslant \|h\|.$$

(ii) Yes, since we can, by a suitable rotation $e^{i\theta}$ achieve that

$$\langle e^{i\theta}g, h\rangle = |\langle g,h\rangle|$$

while $\|g\| = \|e^{i\theta}g\|$.

(iii) Yes. If $D \subset \mathfrak{H}$ is dense and $h \in \mathfrak{H}$ we find a sequence $(d_j)_j \subset D$ with $\lim_j d_j = h$. Since the scalar product and the norm are continuous, we get

$$\lim_j \frac{\langle d_j,h\rangle}{\|d_j\|} = \frac{\langle h,h\rangle}{\|h\|} = \|h\|$$

and we conclude that

$$\|h\| \leqslant \sup_j |\langle d_j/\|d_j\|, h\rangle| \leqslant \sup_{d\in D,\ \|d\|=1}|\langle d,h\rangle|.$$

The reverse inequality is trivial.

Problem 21.9 Let $x, y \in \operatorname{span}\{e_j,\ j\in\mathbb{N}\}$. By definition, there exist numbers $m,n \in \mathbb{N}$ and 'coordinates' $\xi_1,\dots,\xi_m,\eta_1,\dots,\eta_n \in \mathbb{K}$ such that

$$x = \sum_{j=1}^m \xi_j e_j \quad\text{and}\quad y = \sum_{k=1}^n \eta_k e_k.$$

Without loss of generality we can assume that $m \leqslant n$. By defining

$$\xi_{m+1} := 0, \dots, \xi_n := 0$$

we can write for all $\alpha,\beta\in\mathbb{K}$

$$x = \sum_{j=1}^n \xi_j e_j \quad\text{and}\quad y = \sum_{k=1}^n \eta_k e_k \quad\text{and}\quad \alpha x + \beta y = \sum_{\ell=1}^n (\alpha\xi_\ell + \beta\eta_\ell)e_k.$$

This shows that $\operatorname{span}\{e_j,\ j\in\mathbb{N}\} \subset \mathfrak{H}$ is a linear subspace.

Problem 21.10 (i) Since $\sum_{j=1}^{\infty} a_j^2 = \infty$ there is some number $j_1 \in \mathbb{N}$ such that

$$\sum_{j=1}^{j_1} a_j^2 > 1.$$

Since the remaining tail of the series $\sum_{j>j_1} a_j^2 = \infty$ we can construct recursively a strictly increasing sequence $(j_k)_{k\in\mathbb{N}_0} \subset \mathbb{N}$, $j_0 := 1$, such that

$$\sum_{j\in J_k} a_j^2 > 1 \quad \text{where} \quad J_k := (j_k, j_{k+1}] \cap \mathbb{N}.$$

(ii) Define the numbers γ_k as, say,

$$\gamma_k := \frac{1}{k\sqrt{\sum_{j\in J_k} a_j^2}}.$$

Then

$$\begin{aligned}\sum_j b_j^2 &= \sum_k \sum_{j\in J_k} \gamma_k^2 a_j^2 \\ &= \sum_k \gamma_k^2 \sum_{j\in J_k} a_j^2 \\ &= \sum_k \frac{\sum_{j\in J_k} a_j^2}{k^2 \sum_{j\in J_k} a_j^2} \\ &= \sum_k \frac{1}{k^2} < \infty.\end{aligned}$$

Moreover, since

$$\frac{\sum_{j\in J_k} a_j^2}{\sqrt{\sum_{j\in J_k} a_j^2}} \geqslant 1,$$

we get

$$\begin{aligned}\sum_j a_j b_j &= \sum_k \sum_{j\in J_k} \gamma_k a_j^2 \\ &= \sum_k \gamma_k \sum_{j\in J_k} a_j^2 \\ &= \sum_k \frac{1}{k} \frac{\sum_{j\in J_k} a_j^2}{\sqrt{\sum_{j\in J_k} a_j^2}} \\ &\geqslant \sum_k \frac{1}{k} = \infty.\end{aligned}$$

(iii) We want to show (note that we renamed $\beta := a$ and $\alpha := b$ for notational reasons) that for any sequence $\alpha = (\alpha_j)_j$ we have:

$$\forall\, \beta \in \ell^2 \,:\, \langle \alpha, \beta \rangle < \infty \implies \alpha \in \ell^2.$$

Assume, to the contrary, that $\alpha \notin \ell^2$. Then $\sum_j \alpha_j^2 = \infty$ and, by part (i), we can find a sequence of j_k with the properties described in (i). Because of part (ii) there is a sequence $\beta = (\beta_j)_j \in \ell^2$ such that the scalar product $\langle \alpha, \beta \rangle = \infty$. This contradicts our assumption, i.e. α should have been in ℓ^2 in the first place.

(iv) Since, by Theorem 21.15 every separable Hilbert space has a basis $(e_j)_{j\in\mathbb{N}} \subset \mathfrak{H}$, we can identify $h \in \mathfrak{H}$ with the sequence of 'coordinates' $(\langle h, e_j\rangle)_{j\in\mathbb{N}}$ and it is clear that (iii) implies (iv).

Problem 21.11 (i) Since $P^2 = P$ is obvious by the uniqueness of the minimizing element, this part follows already from Remark 21.7.

(ii) Note that for $u, v \in \mathfrak{H}$ we have

$$\forall\, h \in \mathfrak{H} : \langle u, h\rangle = \langle v, h\rangle \implies u = v.$$

Indeed, consider $h := u - v$. Then

$$\langle u, h\rangle = \langle v, h\rangle \implies 0 = \langle u - v, h\rangle = \langle u - v, u - v\rangle = |u - v|^2$$

so that $u = v$.

Linearity of P: Let $\alpha, \beta \in \mathbb{K}$ and $f, g, h \in \mathfrak{H}$. Then

$$\begin{aligned}\langle P(\alpha f + \beta g), h\rangle &= \langle \alpha f + \beta g, Ph\rangle \\ &= \alpha\langle f, Ph\rangle + \beta\langle g, Ph\rangle \\ &= \alpha\langle Pf, h\rangle + \beta\langle Pg, h\rangle \\ &= \langle \alpha Pf + \beta Pg, h\rangle\end{aligned}$$

and we conclude that $P(\alpha f + \beta g) = \alpha Pf + \beta Pg$.

Continuity of P: We have for all $h \in \mathfrak{H}$

$$\|Ph\|^2 = \langle Ph, Ph\rangle = \langle P^2 h, h\rangle = \langle Ph, h\rangle \leqslant \|Ph\| \cdot \|h\|$$

and dividing by $\|Ph\|$ shows that P is continuous, even a contraction.

Closedness of $P(\mathfrak{H})$: Note that $f \in P(\mathfrak{H})$ if, and only if, $f = Ph$ for some $h \in \mathfrak{H}$. Since $P^2 = P$ we get

$$\begin{aligned} f = Ph &\iff f - Ph = 0 \\ &\iff f - P^2 h = 0 \\ &\iff f - Pf = 0 \\ &\iff f \in (\mathrm{id} - P)^{-1}(\{0\})\end{aligned}$$

and since P is continuous and $\{0\}$ is a closed set, $(\mathrm{id} - P)^{-1}(\{0\})$ is closed and the above line shows $P(\mathfrak{H}) = (\mathrm{id} - P)^{-1}(\{0\})$ is closed.

Projection: In view of Corollary 21.6 we have to show that $Ph - h$ is for any $h \in \mathfrak{H}$ orthogonal to $f \in P(\mathfrak{H})$. But

$$\begin{aligned}\langle Ph - h, f\rangle &= \langle Ph, f\rangle - \langle h, f\rangle \\ &= \langle h, Pf\rangle - \langle h, f\rangle \\ &= \langle h, f\rangle - \langle h, f\rangle \;=\; 0.\end{aligned}$$

(iii) Since, by assumption, $\|Ph\| \leqslant \|h\|$, P is continuous and closedness follows just as in (ii). It is, therefore, enough to show that P is an orthogonal projection.

We will show that $\mathcal{N} := \{h \in \mathfrak{H} : Ph = 0\}$ satisfies $\mathcal{N}^\perp = P(\mathfrak{H})$.

For this we observe that if $h \in \mathfrak{H}$, $P(Ph - h) = P^2 h - Ph = Ph - Ph = 0$ so that $Ph - h \in \mathcal{N}$. In particular

$$\begin{aligned} h \in \mathcal{N}^\perp &\implies y = Ph - h \in \mathcal{N} \\ &\implies Ph = h + y \quad \text{with} \quad h \perp y. \end{aligned} \tag{*}$$

Thus,

$$\|h\|^2 + \|y\|^2 = \|Ph\|^2 \leqslant \|h\|^2 \implies \|y\|^2 \implies y = 0.$$

We conclude that

$$h \in \mathcal{N}^\perp \implies Ph - h = 0 \implies Ph = h \implies h \in P(\mathfrak{H})$$

and we have shown that $\mathcal{N}^\perp \subset P(\mathfrak{H})$.

To see the converse direction we pick $h \in P(\mathfrak{H})$ and find $Ph = h$. Since $\mathfrak{H} = \mathcal{N} \oplus \mathcal{N}^\perp$ we have $h = x + x^\perp$ with $x \in \mathcal{N}$ and $x^\perp \in \mathcal{N}^\perp$. Thus,

$$Ph = Px + P(x^\perp) = P(x^\perp) \overset{(*)}{=} x^\perp,$$

thus

$$h = Ph = x^\perp \implies P(\mathfrak{H}) \subset \mathcal{N}^\perp.$$

We have seen that $P(\mathfrak{H}) = \mathcal{N}^\perp \perp \mathcal{N} = \operatorname{kernel}(P)$. This means that

$$\langle Ph - h, Ph \rangle = 0$$

and we conclude that P is an orthogonal projection.

Problem 21.12 (i) Pick $u_j \in Y_j$ and $u_k \in Y_k$, $j \neq k$. Then

$$\begin{aligned} \int_{A_m} u_j u_k \, d\mu &\leqslant \sqrt{\int_{A_m} u_j^2 \, d\mu} \sqrt{\int_{A_m} u_k^2 \, d\mu} \\ &= \begin{cases} 0 \cdot 0 & \text{if } m \notin \{j, k\} \\ \sqrt{\cdots} \cdot 0 & \text{if } m = j, m \neq k \\ 0 \cdot \sqrt{\cdots} & \text{if } m \neq j, m = k \end{cases} \\ &= 0. \end{aligned}$$

(ii) Let $u \in L^2(\mu)$ and set $w_j := w \mathbb{1}_{A_1 \cup \cdots \cup A_j}$. Since $(A_1 \cup \cdots \cup A_j)^c = A_1^c \cap \cdots \cap A_j^c \downarrow \varnothing$ we get by dominated convergence

$$\|u - w_j\|_2^2 = \int_{(A_1 \cup \cdots \cup A_j)^c} u^2 \, d\mu = \int_{A_1^c \cap \cdots \cap A_j^c} u^2 \, d\mu \xrightarrow{j \to \infty} 0.$$

(iii) P is given by $P_j(u) = u\mathbb{1}_{A_j}$. Clearly, $P_j : L^2(\mu) \to Y_j$ is linear and $P^2 = P$, i.e. it is a projection. Orthogonality follows from

$$\langle u - u\mathbb{1}_{A_j}, u\mathbb{1}_{A_j}\rangle = \int u\mathbb{1}_{A_j^c} \cdot u\mathbb{1}_{A_j}\,d\mu = \int u\mathbb{1}_{\emptyset}\,d\mu = 0.$$

Problem 21.13 (i) See Lemma 22.1 in Chapter 24.

(ii) Set $u_n := E^{\mathcal{A}_n}u$. Then

$$u_n = \sum_{j=0}^{n} \alpha_j \cdot \mathbb{1}_{A_j}, \qquad \alpha_j := \frac{1}{\mu(A_j)} \int_{A_j} u\,d\mu, \quad 0 \leqslant j \leqslant n.$$

where $A_0 := (A_1 \cup \dots \cup A_n)^c$ and $1/\infty := 0$. This follows simply from the consideration that u_n, as an element of $L^2(\mathcal{A}_n)$, must be of the form $\sum_{j=0}^{n} \alpha_j \cdot \mathbb{1}_{A_j}$ while the α_j's are calculated as

$$\langle E^{\mathcal{A}_j}u, \mathbb{1}_{A_j}\rangle = \langle u, E^{\mathcal{A}_j}\mathbb{1}_{A_j}\rangle = \langle u, \mathbb{1}_{A_j}\rangle = \int_{A_j} u\,d\mu$$

(resp. $= 0$ if $\mu(A_0) = \infty$) so that, because of disjointness,

$$\alpha_j\mu(A_j) = \Big\langle \sum_{k=0}^{n} \alpha_k \cdot \mathbb{1}_{A_k}, \mathbb{1}_{A_j}\Big\rangle = \langle E^{\mathcal{A}_j}u, \mathbb{1}_{A_j}\rangle = \int_{A_j} u\,d\mu.$$

Clearly this is a linear map and $u_n \in L^2(\mathcal{A}_n)$. Orthogonality follows because all the $A_0, \dots, A_n$ are disjoint so that

$$\begin{aligned}\langle u - u_n, u_n\rangle &= \Big\langle u - \sum_{j=0}^{n} \alpha_j\mathbb{1}_{A_j}, \sum_{k=0}^{n} \alpha_k\mathbb{1}_{A_k}\Big\rangle \\ &= \sum_{j=0}^{n} \int_{A_j} (u - \alpha_j)\alpha_j\,d\mu \\ &= \sum_{j=0}^{n} \left(\alpha_j \int_{A_j} u\,d\mu - \mu(A_j)\alpha_j^2\right) \\ &= \sum_{j=0}^{n} 0 \;=\; 0.\end{aligned}$$

(iii) We have

$$L^2(\mathcal{A}_n)^\perp = \left\{u - \sum_{j=0}^{n} \alpha_j\mathbb{1}_{A_j} = \sum_{j=0}^{n} (u - \alpha_j)\mathbb{1}_{A_j} \;:\; u \in L^2(\mu)\right\}$$

(iv) In view of Remark 17.2 we have to show that

$$\int_{A_j} E^{\mathcal{A}_n}u\,d\mu = \int_{A_j} E^{\mathcal{A}_{n+1}}u\,d\mu, \quad \forall\, A_0, A_1, \dots, A_n.$$

Thus

$$\int_{A_j} E^{\mathcal{A}_n}u\,d\mu = \langle E^{\mathcal{A}_n}u, \mathbb{1}_{A_j}\rangle = \langle u, E^{\mathcal{A}_n}\mathbb{1}_{A_j}\rangle = \langle u, \mathbb{1}_{A_j}\rangle = \int_{A_j} u\,d\mu$$

for all $0 \leqslant j \leqslant n$. The same argument shows also that

$$\int_{A_j} E^{\mathcal{A}_{n+1}}u\,d\mu = \int_{A_j} u\,d\mu \quad \forall\, j = 1, 2, \dots, n.$$

Since the $A_1, A_2, \dots$ are pairwise disjoint and $A_0 = (A_1 \cup \dots \cup A_n)^c$, we have $A_{n+1} \subset A_0$ and $A_j \cap A_0 = \emptyset$, $1 \leqslant j \leqslant n$; if $j = 0$ we get

$$\begin{aligned}
&\int_{A_0} E^{\mathcal{A}_{n+1}} u \, d\mu \\
&= \int_{A_0} \left(\mathbb{1}_{A_{n+1}} \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mathbb{1}_{A_0 \setminus A_{n+1}} \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \right) d\mu \\
&= \mu(A_0 \cap A_{n+1}) \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mu(A_0 \setminus A_{n+1}) \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \\
&= \mu(A_{n+1}) \frac{\int_{A_{n+1}} u \, d\mu}{\mu(A_{n+1})} + \mu(A_0 \setminus A_{n+1}) \frac{\int_{A_0 \setminus A_{n+1}} u \, d\mu}{\mu(A_0 \setminus A_{n+1})} \\
&= \int_{A_{n+1}} u \, d\mu + \int_{A_0 \setminus A_{n+1}} u \, d\mu \\
&= \int_{A_0} u \, d\mu.
\end{aligned}$$

The claim follows.

Remark. It is, actually, better to show that for $u_n := E^{\mathcal{A}_n} u$ the sequence $(u_n^2)_n$ is a sub-Martingale. (The advantage of this is that we do not have to assume that $u \in L^1$ and that $u \in L^2$ is indeed enough....). O.k.:

We have

$$\begin{aligned}
A_0^n &:= (A_1 \uplus \dots \uplus A_n)^c = A_1^c \cap \dots \cap A_n^c \\
A_0^{n+1} &:= (A_1 \uplus \dots \uplus A_n \uplus A_{n+1})^c = A_0^n \cap A_{n+1}^c
\end{aligned}$$

and

$$\begin{aligned}
E^{\mathcal{A}_n} u &= \sum_{j=1}^{n} \mathbb{1}_{A_j} \int_{A_j} u \frac{d\mu}{\mu(A_j)} + \mathbb{1}_{A_0^n} \int_{A_0^n} u \frac{d\mu}{\mu(A_0^n)} \\
E^{\mathcal{A}_{n+1}} u &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \int_{A_j} u \frac{d\mu}{\mu(A_j)} + \mathbb{1}_{A_0^{n+1}} \int_{A_0^{n+1}} u \frac{d\mu}{\mu(A_0^{n+1})}
\end{aligned}$$

with the convention that $1/\infty = 0$. Since the A_j's are mutually disjoint,

$$\begin{aligned}
\left(E^{\mathcal{A}_n} u\right)^2 &= \sum_{j=1}^{n} \mathbb{1}_{A_j} \left[\int_{A_j} u \frac{d\mu}{\mu(A_j)} \right]^2 + \mathbb{1}_{A_0^n} \left[\int_{A_0^n} u \frac{d\mu}{\mu(A_0^n)} \right]^2 \\
\left(E^{\mathcal{A}_{n+1}} u\right)^2 &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \left[\int_{A_j} u \frac{d\mu}{\mu(A_j)} \right]^2 + \mathbb{1}_{A_0^{n+1}} \left[\int_{A_0^{n+1}} u \frac{d\mu}{\mu(A_0^{n+1})} \right]^2.
\end{aligned}$$

We have to show that $\left(E^{\mathcal{A}_n} u\right)^2 = u_n^2 \leqslant u_{n+1}^2 = \left(E^{\mathcal{A}_{n+1}} u\right)^2$. If $\mu(A_0^{n+1}) = \infty$ this follows trivially since in this case

$$\begin{aligned}
\left(E^{\mathcal{A}_n} u\right)^2 &= \sum_{j=1}^{n} \mathbb{1}_{A_j} \left[\int_{A_j} u \frac{d\mu}{\mu(A_j)} \right]^2 \\
\left(E^{\mathcal{A}_{n+1}} u\right)^2 &= \sum_{j=1}^{n+1} \mathbb{1}_{A_j} \left[\int_{A_j} u \frac{d\mu}{\mu(A_j)} \right]^2.
\end{aligned}$$

If $\mu(A_0^{n+1}) < \infty$ we get

$$\begin{aligned}
&\left(E^{\mathcal{A}_n}u\right)^2 - \left(E^{\mathcal{A}_{n+1}}u\right)^2 \\
&= \mathbb{1}_{A_0^n}\left[\int_{A_0^n} u\,\frac{d\mu}{\mu(A_0^n)}\right]^2 - \mathbb{1}_{A_{n+1}}\left[\int_{A_{n+1}} u\,\frac{d\mu}{\mu(A_{n+1})}\right]^2 \\
&\qquad + \mathbb{1}_{A_0^{n+1}}\left[\int_{A_0^{n+1}} u\,\frac{d\mu}{\mu(A_0^{n+1})}\right]^2 \\
&= \mathbb{1}_{A_{n+1}}\left(\left[\int_{A_{n+1}} u\,\frac{d\mu}{\mu(A_0^n)}\right]^2 - \left[\int_{A_{n+1}} u\,\frac{d\mu}{\mu(A_{n+1})}\right]^2\right) \\
&\qquad + \mathbb{1}_{A_0^{n+1}}\left(\left[\int_{A_0^{n+1}} u\,\frac{d\mu}{\mu(A_0^n)}\right]^2 - \left[\int_{A_0^{n+1}} u\,\frac{d\mu}{\mu(A_0^{n+1})}\right]^2\right)
\end{aligned}$$

and each of the expressions in the brackets is negative since

$$A_0^n \supset A_{n+1} \implies \mu(A_0^n) \geqslant \mu(A_{n+1}) \implies \frac{1}{\mu(A_0^n)} \leqslant \frac{1}{\mu(A_{n+1})}$$

and

$$A_0^n \supset A_0^{n+1} \implies \mu(A_0^n) \geqslant \mu(A_0^{n+1}) \implies \frac{1}{\mu(A_0^n)} \leqslant \frac{1}{\mu(A_0^{n+1})}.$$

(v) Set $u_n := E^{\mathcal{A}_n}u$. Since $(u_n)_n$ is a martingale, u_n^2 is a submartingale. In fact, $(u_n^2)_n$ is even uniformly integrable. For this we remark that

$$u_n = \sum_{j=1}^n \mathbb{1}_{A_j}\int_{A_j} u(x)\,\frac{\mu(dx)}{\mu(A_j)} + \mathbb{1}_{A_0^n}\int_{A_0^n} u(x)\,frac\mu(dx)\mu(A_0^n)$$

$(1/\infty := 0)$ and that the function

$$v := \sum_{j=1}^\infty \mathbb{1}_{A_j}\int_{A_j} u(x)\,\frac{\mu(dx)}{\mu(A_j)}$$

is in $L^2(\mathcal{A}_\infty)$. Only integrability is a problem: since the A_j's are mutually disjoint, the square of the series defining v factorizes, i.e.

$$\begin{aligned}
\int v^2(y)\,\mu(dy) &= \int\left(\sum_{j=1}^\infty \mathbb{1}_{A_j}(y)\int_{A_j} u(x)\,\frac{\mu(dx)}{\mu(A_j)}\right)^2\mu(dy) \\
&= \sum_{j=1}^\infty \int \mathbb{1}_{A_j}(y)\,\mu(dy)\left(\int_{A_j} u(x)\,\frac{\mu(dx)}{\mu(A_j)}\right)^2 \\
&\leqslant \sum_{j=1}^\infty \int \mathbb{1}_{A_j}(y)\,\mu(dy)\int_{A_j} u^2(x)\,\frac{\mu(dx)}{\mu(A_j)} \\
&= \sum_{j=1}^\infty \int_{A_j} u^2(x)\,\mu(dx) \\
&= \int u^2(x)\,\mu(dx)
\end{aligned}$$

where we used Beppo Levi's theorem (twice) and Jensen's inequality. In fact,

$$v = E^{\mathcal{A}_\infty}u.$$

Since $u_n(x) = v(x)$ for all $x \in A_1 \cup \dots \cup A_n$, and since $A_0^n = (A_1 \cup \dots \cup A_n)^c \in \mathcal{A}_n$ we find by the submartingale property

$$\begin{aligned}\int_{\{u_n^2 > (2v)^2\}} u_n^2 \, d\mu &\leqslant \int_{A_0^n} u_n^2 \, d\mu \\ &\leqslant \int_{A_0^n} u^2 \, d\mu \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

by dominated convergence since $A_0^n \to \varnothing$ and $u^2 \in L^1(\mu)$.

Using the convergence theorem for UI (sub)martingales, Theorem 18.6, we conclude that u_j^2 converges pointwise and in L^1-sense to some $u_\infty^2 \in L^1(\mathcal{A}_\infty)$ and that $(u_j^2)_{j \in \mathbb{N} \cup \{\infty\}}$ is again a submartingale. By Riesz' convergence theorem 12.10 we conclude that $u_j \to u_\infty$ in L^2-norm.

Remark: We can also identify u_∞ with v: since $E^{\mathcal{A}_j} v = u_j = E^{\mathcal{A}_j} u_\infty$ it follows that for $k = 1, 2, \dots, j$ and all j

$$0 = \langle E^{\mathcal{A}_j} v - E^{\mathcal{A}_j} u_\infty, \mathbb{1}_{A_k} \rangle = \langle v - u_\infty, E^{\mathcal{A}_j} \mathbb{1}_{A_k} \rangle = \langle v - u_\infty, \mathbb{1}_{A_k} \rangle$$

i.e. $v = u_\infty$ on all sets of the $\cap$-stable generator of $\mathcal{A}_\infty$ which can easily be extended to contain an exhausting sequence $A_1 \uplus \dots \uplus A_n$ of sets of finite μ-measure.

(vi) The above considerations show that the functions

$$D := \left\{ \alpha_0 \mathbb{1}_{A_0^n} + \sum_{j=1}^n \alpha_j \mathbb{1}_{A_j} \;:\; n \in \mathbb{N}, \alpha_j \in \mathbb{R} \right\}$$

(if $\mu(A_0^n) = \infty$, then $\alpha_0 = 0$) are dense in $L^2(\mathcal{A}_\infty)$. It is easy to see that

$$E := \left\{ q_0 \mathbb{1}_{A_0^n} + \sum_{j=1}^n q_j \mathbb{1}_{A_j} \;:\; n \in \mathbb{N}, \alpha_j \in \mathbb{Q} \right\}$$

(if $\mu(A_0^n) = \infty$, then $q_0 = 0$) is countable and dense in D so that the claim follows.

22 Conditional expectations in L^2 Solutions to Problems 22.1–22.3

Problem 22.1 In Theorem 22.4(vii) we have seen that

$$\mathbb{E}^{\mathcal{H}}\,\mathbb{E}^{\mathcal{G}}\,u = \mathbb{E}^{\mathcal{H}}\,u.$$

Since, by 22.4(i) and 22.1 $\mathbb{E}^{\mathcal{H}}\,u \in L^2(\mathcal{H}) \subset L^2(\mathcal{G})$ we have, because of 22.4

$$\mathbb{E}^{\mathcal{G}}\,\mathbb{E}^{\mathcal{H}}\,u = \mathbb{E}^{\mathcal{H}}\,u.$$

Problem 22.2 Note that, since $\mathbb{E}^{\mathcal{G}}$ is (currently...) only defined for L^2-functions the problem implicitly requires that $f \in L^2(\mathcal{A},\mu)$. (A look at the next section reveals that this is not really necessary...). Below we will write $\langle\bullet,\bullet\rangle_{L^2(\mu)}$ resp. $\langle\bullet,\bullet\rangle_{L^2(\nu)}$ to indicate which scalar product is meant.

We begin with a general consideration: Let u, w be functions such that $u^2, v^2 \in L^2(\mu)$. Then we have $|u\cdot w| \leqslant \frac{1}{2}(u^2+w^2) \in L^2(\mu)$ and, using again the elementary inequality

$$|xy| \leqslant \frac{x^2}{2} + \frac{y^2}{2}$$

for $x = |u|/\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(u^2)}$ and $y = |w|/\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}$ we conclude that on $G_n := \{\mathbb{E}^{\mathcal{G}}_\mu(u^2) > \frac{1}{n}\} \cap \{\mathbb{E}^{\mathcal{G}}_\mu(w^2) > \frac{1}{n}\}$

$$\frac{|u|\cdot|w|}{\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}}\,\mathbb{1}_{G_n} \leqslant \left[\frac{u^2}{2\,\mathbb{E}^{\mathcal{G}}_\mu(u^2)} + \frac{w^2}{2\,\mathbb{E}^{\mathcal{G}}_\mu(w^2)}\right]\mathbb{1}_{G_n}.$$

Taking conditional expectations on both sides yields, since $G_n \in \mathcal{G}$:

$$\frac{\mathbb{E}^{\mathcal{G}}_\mu\left(|u|\cdot|w|\right)}{\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}}\,\mathbb{1}_{G_n} \leqslant \mathbb{1}_{G_n}.$$

Multiplying through with the denominator of the lhS and letting $n \to \infty$ gives

$$\left|\mathbb{E}^{\mathcal{G}}_\mu(uw)\right|\mathbb{1}_{G^*} \leqslant \mathbb{E}^{\mathcal{G}}_\mu\left(|uw|\right)\mathbb{1}_{G^*} \leqslant \sqrt{\mathbb{E}^{\mathcal{G}}_\mu(u^2)}\sqrt{\mathbb{E}^{\mathcal{G}}_\mu(w^2)}$$

on the set $G^* := G_u \cap G_w := \{\mathbb{E}^{\mathcal{G}}_\mu\,u^2 > 0\} \cap \{\mathbb{E}^{\mathcal{G}}_\mu\,w^2 > 0\}$.

(i) Set $G^* := \{\mathbb{E}^{\mathcal{G}}_\mu\,f > 0\}$ and $G_n := \{\mathbb{E}^{\mathcal{G}}_\mu\,f > \frac{1}{n}\}$. Clearly, using the Markov inequality,

$$\mu(G_n) \leqslant n^2 \int (\mathbb{E}^{\mathcal{G}}_\mu\,f)^2\,d\mu \leqslant n \int f^2\,d\mu < \infty$$

so that by monotone convergence we find for all $G \in \mathcal{G} \cap G^*$

$$\begin{aligned}
\nu(G) &= \langle f, \mathbb{1}_G \rangle_{L^2(\mu)} \\
&= \sup_n \langle f, \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \sup_n \langle f, \mathbb{E}_\mu^{\mathcal{G}} \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \sup_n \langle \mathbb{E}_\mu^{\mathcal{G}} f, \mathbb{1}_{G \cap G_n} \rangle_{L^2(\mu)} \\
&= \langle \mathbb{E}_\mu^{\mathcal{G}} f, \mathbb{1}_G \rangle_{L^2(\mu)}
\end{aligned}$$

which means that $\nu|_{\mathcal{G} \cap G^*} = \mathbb{E}^{\mathcal{G}} f \cdot \mu|_{\mathcal{G} \cap G^*}$.

Following the hint, we define for *bounded* $u \in L^2(\nu)$

$$\mathbb{E}_\nu^{\mathcal{G}} u := \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*}.$$

Let us show that $\mathbb{E}_\nu^{\mathcal{G}} u \in L^2(\nu)$. Set $G_{\sqrt{f}u} := \{\mathbb{E}_\mu^{\mathcal{G}}(f \cdot u^2) > 0\}$. Then, for *bounded* $u \in L^2(\nu)$

$$\begin{aligned}
&\left\| \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \right\|_{L^2(\nu)}^2 \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\left[\mathbb{E}_\mu^{\mathcal{G}}(fu)\right]^2}{\left[\mathbb{E}_\mu^{\mathcal{G}} f\right]^2} \, d\nu \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\left[\mathbb{E}_\mu^{\mathcal{G}}(fu)\right]^2}{\left[\mathbb{E}_\mu^{\mathcal{G}} f\right]^2} f \, d\mu \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\left[\mathbb{E}_\mu^{\mathcal{G}}(fu)\right]^2}{\left[\mathbb{E}_\mu^{\mathcal{G}} f\right]^2} \mathbb{E}_\mu^{\mathcal{G}} f \, d\mu \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\left[\mathbb{E}_\mu^{\mathcal{G}}(fu)\right]^2}{\mathbb{E}_\mu^{\mathcal{G}} f} \, d\mu \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\left[\mathbb{E}_\mu^{\mathcal{G}}\left[\sqrt{f}(\sqrt{f}u)\right]\right]^2}{\mathbb{E}_\mu^{\mathcal{G}} f} \, d\mu \\
&\leqslant \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \frac{\mathbb{E}_\mu^{\mathcal{G}} f \cdot \mathbb{E}_\mu^{\mathcal{G}}\left[fu^2\right]}{\mathbb{E}_\mu^{\mathcal{G}} f} \, d\mu \\
&= \int_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \mathbb{E}_\mu^{\mathcal{G}}\left[fu^2\right] d\mu \\
&= \sup_n \int \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} \mathbb{E}_\mu^{\mathcal{G}}\left[fu^2\right] d\mu \\
&= \sup_n \int \mathbb{E}_\mu^{\mathcal{G}} \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} fu^2 \, d\mu \\
&= \sup_n \int \mathbb{1}_{G_n \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} fu^2 \, d\mu \\
&= \int \mathbb{1}_{G^* \cap G_{\sqrt{f}u} \cap G_{\sqrt{f}}} fu^2 \, d\mu \\
&\leqslant \int fu^2 \, d\mu = \left\|\sqrt{f}u\right\|_{L^2(\mu)}^2 = \|u\|_{L^2(\nu)}^2 < \infty.
\end{aligned}$$

Still for bounded $u \in L^2(\nu)$,

$$
\begin{aligned}
&\int_{G_n \cap \{f<n\} \cap \{\mathbb{E}^{\mathcal{G}}(fu^2)=0\}} \mathbb{E}_\mu^{\mathcal{G}}(fu)\,d\mu \\
&= \int_{G_n \cap \{\mathbb{E}^{\mathcal{G}}(\sqrt{f}u)=0\}} fu\,d\mu \\
&\leqslant \sqrt{\int_{G_n \cap \{f<n\}} f\,d\mu}\sqrt{\int_{G_n \cap \{\mathbb{E}^{\mathcal{G}}(fu^2)=0\}} fu^2\,d\mu} \\
&= \sqrt{\int_{G_n \cap \{f<n\}} f\,d\mu}\sqrt{\int_{G_n \cap \{\mathbb{E}^{\mathcal{G}}(fu^2)=0\}} \mathbb{E}_\mu^{\mathcal{G}} fu^2\,d\mu} \\
&= 0
\end{aligned}
$$

and, using monotone convergence, we have

$$
\left\| \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\|_{L^2(\nu)}^2 \leqslant \|u\|_{L^2(\nu)}^2
$$

for all bounded $u \in L^2(\nu)$, hence — through extension by continuity — for all $u \in L^2(\nu)$. Since

$$
\begin{aligned}
&\left\langle u - \mathbb{E}_\nu^{\mathcal{G}} u, \mathbb{E}_\nu^{\mathcal{G}} u \right\rangle_{L^2(\nu)} \\
&= \left\langle fu - f\,\mathbb{E}_\nu^{\mathcal{G}} u, \mathbb{E}_\nu^{\mathcal{G}} u \right\rangle_{L^2(\mu)} \\
&= \left\langle fu - f\frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathcal{G}}\left[fu - f\frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*}\right], \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathcal{G}}(fu) - \mathbb{E}_\mu^{\mathcal{G}}\left[f\frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*}\right], \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathcal{G}}(fu) - \mathbb{E}_\mu^{\mathcal{G}}(f)\,\frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= \left\langle \mathbb{E}_\mu^{\mathcal{G}}(fu) - \mathbb{E}_\mu^{\mathcal{G}}(fu)\,\mathbb{1}_{G^*}, \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} \right\rangle_{L^2(\mu)} \\
&= 0
\end{aligned}
$$

which shows that $\mathbb{E}_\nu^{\mathcal{G}}$ is the (uniquely determined) orthogonal projection onto the space $L^2(\nu, \mathcal{G})$.

(Note that we have, implicitly, extended $\mathbb{E}_\mu^{\mathcal{G}}$ onto L^1....)

(ii) The condition that $f\mathbb{1}_{G^*}$ is $\mathcal{G}$-measurable will do. Indeed, since $G^* \in \mathcal{G}$:

$$
\mathbb{E}_\nu^{\mathcal{G}} u = \frac{\mathbb{E}_\mu^{\mathcal{G}}(fu)}{\mathbb{E}_\mu^{\mathcal{G}} f} \mathbb{1}_{G^*} = \frac{\mathbb{E}_\mu^{\mathcal{G}}((f\mathbb{1}_{G^*})u)}{\mathbb{E}_\mu^{\mathcal{G}}(f\mathbb{1}_{G^*})} = \frac{(f\mathbb{1}_{G^*})\,\mathbb{E}_\mu^{\mathcal{G}}(u)}{(f\mathbb{1}_{G^*})} = \mathbb{E}_\mu^{\mathcal{G}} u.
$$

In fact, if $f \in L^4(\mu, \mathcal{A})$ this is also necessary:

$$
\mathbb{E}_\mu^{\mathcal{G}} f = \mathbb{E}_\nu^{\mathcal{G}} f
$$

implies, because of (i), that

$$\mathbb{E}_\mu^{\mathcal{G}} f = \frac{\mathbb{E}_\mu^{\mathcal{G}}(f^2)}{\mathbb{E}_\mu^{\mathcal{G}} f}\,\mathbb{1}_{\{\mathbb{E}_\mu^{\mathcal{G}} f>0\}} \iff \big(\mathbb{E}_\mu^{\mathcal{G}} f\big)^2 = \mathbb{E}_\mu^{\mathcal{G}}(f^2)\,\mathbb{1}_{\{\mathbb{E}_\mu^{\mathcal{G}} f>0\}}$$
$$\iff \big(\mathbb{E}_\mu^{\mathcal{G}} f\big)^2 = \mathbb{E}_\mu^{\mathcal{G}}(f^2).$$

Thus,

$$\mathbb{E}_\mu^{\mathcal{G}}\Big[\big(f - \mathbb{E}_\mu^{\mathcal{G}} f\big)^2\Big] = 0,$$

which means that on the set $G^* = \bigcup_n G_n$ with $\mu(G_n) < \infty$, see above,

$$0 = \int_{G_n} \mathbb{E}_\mu^{\mathcal{G}}(f - \mathbb{E}_\mu^{\mathcal{G}} f)^2\,d\mu = \int_{G_n} (f - \mathbb{E}_\mu^{\mathcal{G}} f)^2\,d\mu$$

i.e. $f = \mathbb{E}_\mu^{\mathcal{G}} f$ on $G^* = \{\mathbb{E}_\mu^{\mathcal{G}} f > 0\}$

Problem 22.3 Since $\mathcal{G} = \{G_1, \dots, G_n\}$ such that the G_j's form a mutually disjoint partition of the whole space X, we have

$$L^2(\mathcal{G}) = \left\{\sum_{j=1}^n \alpha_j\,\mathbb{1}_{G_j} \;:\; \alpha_j \in \mathbb{R}\right\}.$$

It is, therefore, enough to determine the values of the α_j. Using the symmetry and idempotency of the conditional expectation we get for $k \in \{1, 2, \dots, n\}$

$$\langle \mathbb{E}^{\mathcal{G}} u, \mathbb{1}_{G_k}\rangle = \langle u, \mathbb{E}^{\mathcal{G}}\,\mathbb{1}_{G_k}\rangle = \langle u, \mathbb{1}_{G_k}\rangle = \int_{G_k} u\,d\mu.$$

On the other hand, using that $\mathbb{E}^{\mathcal{G}} u \in L^2(\mathcal{G})$ we find

$$\langle \mathbb{E}^{\mathcal{G}} u, \mathbb{1}_{G_k}\rangle = \Big\langle \sum_{j=1}^n \alpha_j\,\mathbb{1}_{G_j}, \mathbb{1}_{G_k}\Big\rangle = \sum_{j=1}^n \alpha_j \langle \mathbb{1}_{G_j}, \mathbb{1}_{G_k}\rangle = \alpha_k \mu(G_k)$$

and we conclude that

$$\alpha_k = \frac{1}{\mu(G_k)} \int_{G_k} u\,d\mu = \int_{G_k} u(x)\,\frac{\mu(dx)}{\mu(G_k)}\,.$$

23 Conditional expectations in L^p Solutions to Problems 23.1–23.12

Problem 23.1 In this problem it is helpful to keep the distinction between $\mathbb{E}^{\mathcal{G}}$ defined on $L^2(\mathcal{A})$ and the extension $E^{\mathcal{G}}$ defined on $L^{\mathcal{G}}(\mathcal{A})$.

Since $\mu|_{\mathcal{A}}$ is σ-finite we can find an exhausting sequence of sets $A_n \uparrow X$ with $\mu(A_n) < \infty$. Setting for $u, w \in L^{\mathcal{G}}(\mathcal{A})$ with $uE^{\mathcal{G}}w \in L^1(\mathcal{A})$ $u_n := \big((-n) \vee u \wedge n\big) \cdot \mathbb{1}_{A_n}$ and $w_n := \big((-n) \vee w \wedge n\big) \cdot \mathbb{1}_{A_n}$ we have found approximating sequences such that $u_n, w_n \in L^1(\mathcal{A}) \cap L^\infty(\mathcal{A})$ and, in particular, $\in L^2(\mathcal{A})$.

(iii): For $u, w \geqslant 0$ we find by monotone convergence, using the properties listed in Theorem 22.4:

$$\begin{aligned}
\langle E^{\mathcal{G}}u, w\rangle &= \lim_n \langle \mathbb{E}^{\mathcal{G}}\, u_n, w\rangle \\
&= \lim_n \lim_m \langle \mathbb{E}^{\mathcal{G}}\, u_n, w_m\rangle \\
&= \lim_n \lim_m \langle u_n, \mathbb{E}^{\mathcal{G}}\, w_m\rangle \\
&= \lim_n \langle u_n, E^{\mathcal{G}}w\rangle \\
&= \langle u, E^{\mathcal{G}}w\rangle.
\end{aligned}$$

In the general case we write

$$\langle E^{\mathcal{G}}u, w\rangle = \langle E^{\mathcal{G}}u^+, w^+\rangle - \langle E^{\mathcal{G}}u^-, w^+\rangle - \langle E^{\mathcal{G}}u^+, w^-\rangle + \langle E^{\mathcal{G}}u^-, w^-\rangle$$

and consider each term separately.

The equality $\langle E^{\mathcal{G}}u, w\rangle = \langle E^{\mathcal{G}}u, E^{\mathcal{G}}w\rangle$ follows similarly.

(iv): we have

$$u = w \implies u_j = w_j \quad \forall\, j \implies \mathbb{E}^{\mathcal{G}}\, u_j = \mathbb{E}^{\mathcal{G}}\, w_j \quad \forall\, j$$

and we get

$$E^{\mathcal{G}}u = \lim_j \mathbb{E}^{\mathcal{G}}\, u_j = \lim_j \mathbb{E}^{\mathcal{G}}\, w_j = w.$$

(ix): we have

$$0 \leqslant u \leqslant 1 \implies 0 \leqslant u_n \leqslant 1 \quad \forall\, n$$

$$\implies 0 \leqslant \mathbb{E}^{\mathcal{G}} u_n \leqslant 1 \quad \forall n$$
$$\implies 0 \leqslant E^{\mathcal{G}} u = \lim_n \mathbb{E}^{\mathcal{G}} u_n \leqslant 1.$$

(x):

$$u \leqslant w \implies 0 \leqslant w - u \implies 0 \leqslant E^{\mathcal{G}}(u - w) = E^{\mathcal{G}} u - E^{\mathcal{G}} w.$$

(xi):

$$\pm u \leqslant |u| \implies \pm E^{\mathcal{G}} u \leqslant E^{\mathcal{G}} |u| \implies \left|E^{\mathcal{G}} u\right| \leqslant E^{\mathcal{G}} |u|.$$

Problem 23.2 Assume first that $\mu|_{\mathcal{G}}$ is σ-finite and denote by $G_k \in \mathcal{G}$, $G_k \uparrow X$ and $\mu(G_k) < \infty$ an exhausting sequence. Then $\mathbb{1}_{G_k} \in L^2(\mathcal{G})$, $\mathbb{1}_{G_k} \uparrow 1$ and

$$E^{\mathcal{G}} 1 = \sup_k \mathbb{E}^{\mathcal{G}} \mathbb{1}_{G_k} = \sup_k \mathbb{1}_{G_k} = 1.$$

Conversely, let $E^{\mathcal{G}} 1 = 1$. Because of Lemma 23.2 there is a sequence $(u_k)_k \subset L^2(\mathcal{A})$ with $u_k \uparrow 1$. By the very definition of $E^{\mathcal{G}}$ we have

$$E^{\mathcal{G}} 1 = \sup_k \mathbb{E}^{\mathcal{G}} u_k = 1,$$

i.e. there is a sequence $g_k := \mathbb{E}^{\mathcal{G}} u_k \in L^2(\mathcal{G})$ such that $g_k \uparrow 1$. Set $G_k := \{g_k > 1 - 1/k\}$ and observe that $G_k \uparrow X$ as well as

$$\begin{aligned}\mu(G_k) &\leqslant \frac{1}{(1-\frac{1}{k})^2} \int g_k^2 \, d\mu \\ &= \frac{1}{(1-\frac{1}{k})^2} \| \mathbb{E}^{\mathcal{G}} u_k \|_{L^2}^2 \\ &\leqslant \frac{1}{(1-\frac{1}{k})^2} \| u_k \|_{L^2}^2 \\ &< \infty.\end{aligned}$$

This shows that $\mu|_{\mathcal{G}}$ is σ-finite.

If $\mathcal{G}$ is not σ-finite, e.g. if $\mathcal{G} = \{\emptyset, G, G^c, X\}$ where $\mu(G) < \infty$ and $\mu(G^c) = \infty$ we find that

$$L^2(\mathcal{G}) = \{c\mathbb{1}_G \,:\, c \in \mathbb{R}\}$$

which means that $E^{\mathcal{G}} 1 = \mathbb{1}_G$ since for every $A \subset G^c$, $A \in \mathcal{A}$ and $\mu(A) < \infty$ we find

$$E^{\mathcal{G}} \mathbb{1}_{A \cup G} = E^{\mathcal{G}}(\mathbb{1}_A + \mathbb{1}_G) = E^{\mathcal{G}} \mathbb{1}_A + E^{\mathcal{G}} \mathbb{1}_G = E^{\mathcal{G}} \mathbb{1}_A + \mathbb{1}_G$$

Since this must be an element of $L^2(\mathcal{G})$, we have necessarily $E^{\mathcal{G}} \mathbb{1}_A = c\mathbb{1}_G$ or

$$\langle c\mathbb{1}_G, \mathbb{1}_G \rangle = \langle E^{\mathcal{G}} \mathbb{1}_A, \mathbb{1}_G \rangle = \langle \mathbb{1}_A, E^{\mathcal{G}} \mathbb{1}_G \rangle = \langle \mathbb{1}_A, \mathbb{1}_G \rangle = \mu(A \cap G) = 0,$$

hence $c = 0$ or $E^{\mathcal{G}} \mathbb{1}_A = 0$.

This shows that

$$E^{\mathcal{G}} 1 = \mathbb{1}_G \leqslant 1$$

is best possible.

Problem 23.3 For this problem it is helpful to distinguish between $\mathbb{E}^{\mathcal{G}}$ (defined on L^2) and the extension $E^{\mathcal{G}}$.

Without loss of generality we may assume that $g \geqslant 0$—otherwise we would consider positive and negative parts separately. Since $g \in L^p(\mathcal{G})$ we have that

$$\mu\{g > 1/j\} \leqslant j^p \int g^p \, d\mu < \infty$$

which means that the sequence $g_j := (j \wedge g)\mathbb{1}_{\{g>1/j\}} \in L^2(\mathcal{G})$. Obviously, $g_j \uparrow g$ pointwise as well as in L^p-sense. Using the results from Theorem 22.4 we get

$$\mathbb{E}^{\mathcal{G}} g_j = g_j \implies E^{\mathcal{G}} = \sup_j \mathbb{E}^{\mathcal{G}} g_j = \sup_j g_j = g.$$

Problem 23.4 For this problem it is helpful to distinguish between $\mathbb{E}^{\mathcal{G}}$ (defined on L^2) and the extension $E^{\mathcal{G}}$.

For $u \in L^p(\mathcal{A})$ we get $E^{\mathcal{H}} E^{\mathcal{G}} u = E^{\mathcal{H}} u$ because of Theorem 23.8(vi) while the other equality $E^{\mathcal{G}} E^{\mathcal{H}} u = E^{\mathcal{H}} u$ follows from Problem 23.3.

If $u \in M^+(\mathcal{A})$ (mind the misprint in the problem!) we get a sequence $u_j \uparrow u$ of functions $u_j \in L^2_+(\mathcal{A})$. From Theorem 22.4 we know that $\mathbb{E}^{\mathcal{G}} u_j \in L^2(\mathcal{G})$ increases and, by definition, it increases towards $E^{\mathcal{G}} u$. Thus,

$$\mathbb{E}^{\mathcal{H}} \mathbb{E}^{\mathcal{G}} u_j = \mathbb{E}^{\mathcal{H}} u_j \uparrow E^{\mathcal{H}} u$$

while

$$\mathbb{E}^{\mathcal{H}} \mathbb{E}^{\mathcal{G}} u_j \uparrow E^{\mathcal{H}}\big(\sup_j \mathbb{E}^{\mathcal{G}} u_j\big) = E^{\mathcal{H}} E^{\mathcal{G}} u.$$

The other equality is similar.

Problem 23.5 We know that

$$L^p(\mathcal{A}_n) = \left\{ \sum_{j=1}^n c_j \mathbb{1}_{[j-1,j)} \;:\; c_j \in \mathbb{R} \right\}$$

since $c_0 \mathbb{1}_{[n,\infty)} \in L^p$ if, and only if, $c_0 = 0$. Thus, $E^{\mathcal{A}_n} u$ is of the form

$$E^{\mathcal{A}_n} u(x) = \sum_{j=1}^n c_j \mathbb{1}_{[j-1,j)}(x)$$

and integrating over $[k-1,k)$ yields

$$\int_{[k-1,k)} E^{\mathcal{A}_n} u(x) \, dx = c_k.$$

Since

$$\begin{aligned}\int_{[k-1,k)} E^{\mathcal{A}_n} u(x) \, dx &= \langle E^{\mathcal{A}_n} u, \mathbb{1}_{[k-1,k)} \rangle \\ &= \langle u, E^{\mathcal{A}_n} \mathbb{1}_{[k-1,k)} \rangle\end{aligned}$$

$$= \langle u, \mathbb{1}_{[k-1,k)} \rangle$$
$$= \int_{[k-1,k)} u(x)\,dx$$

we get

$$E^{\mathcal{A}_n} u(x) = \sum_{j=1}^{n} \int_{[j-1,j)} u(t)\,dt \mathbb{1}_{[j-1,j)}(x).$$

Problem 23.6 For this problem it is helpful to distinguish between $\mathbb{E}^{\mathcal{G}}$ (defined on L^2) and the extension $E^{\mathcal{G}}$.

If $\mu(X) = \infty$ and if $\mathcal{G} = \{\varnothing, X\}$, then $L^1(\mathcal{G}) = \{0\}$ which means that $E^{\mathcal{G}} u = 0$ for any $u \in L^1(\mathcal{A})$. Thus for integrable functions $u > 0$ and $\mu|_{\mathcal{G}}$ not σ-finite we can only have '$\leqslant$'.

If $\mu|_{\mathcal{G}}$ is σ-finite and if $G_j \uparrow X$, $G_j \in \mathcal{G}$, $\mu(G_j) < \infty$ is an exhausting sequence, we find for any $u \in L^1_+(\mathcal{A})$

$$\begin{aligned}
\int E^{\mathcal{G}} u\,d\mu &= \sup_j \int_{G_j} E^{\mathcal{G}} u\,d\mu \\
&= \sup_j \langle E^{\mathcal{G}} u, \mathbb{1}_{G_j} \rangle \\
&= \sup_j \langle u, E^{\mathcal{G}} \mathbb{1}_{G_j} \rangle \\
&= \sup_j \langle u, \mathbb{1}_{G_j} \rangle \\
&= \langle u, 1 \rangle \\
&= \int u\,d\mu.
\end{aligned}$$

If $\mu|_{\mathcal{G}}$ is not σ-finite and if $u \geqslant 0$, we perform a similar calculation with an exhausting sequence $A_j \in \mathcal{A}$, $A_j \uparrow X$, $\mu(A_j) < \infty$ (it is implicit that $\mu|_{\mathcal{A}}$ is σ-finite as otherwise the conditional expectation would not be defined!):

$$\begin{aligned}
\int E^{\mathcal{G}} u\,d\mu &= \sup_j \int_{A_j} E^{\mathcal{G}} u\,d\mu \\
&= \sup_j \langle E^{\mathcal{G}} u, \mathbb{1}_{A_j} \rangle \\
&= \sup_j \langle u, E^{\mathcal{G}} \mathbb{1}_{A_j} \rangle \\
&\leqslant \langle u, 1 \rangle \\
&= \int u\,d\mu.
\end{aligned}$$

Problem 23.7

Proof of Corollary 23.11: Since

$$\liminf_{j\to\infty} u_j = \sup_k \inf_{j\geqslant k} u_j$$

we get

$$\mathbb{E}^{\mathcal{G}} \big(\inf_{j\geqslant k} u_j \big) \leqslant \mathbb{E}^{\mathcal{G}} u_m \quad \forall\, m \geqslant k$$

thus

$$\mathbb{E}^{\mathcal{G}}\big(\inf_{j\geqslant k} u_j\big) \leqslant \inf_{m\geqslant k}\mathbb{E}^{\mathcal{G}} u_m \leqslant \sup_k \inf_{m\geqslant k}\mathbb{E}^{\mathcal{G}} u_m = \liminf_{m\to\infty}\mathbb{E}^{\mathcal{G}} u_m.$$

Since on the other hand the sequence $\inf_{j\geqslant k} u_j$ increases, as $k\to\infty$, towards $\sup_k\inf_{j\geqslant k} u_j$ we can use the conditional Beppo Levi theorem 23.10 on the left-hand side and find

$$\mathbb{E}^{\mathcal{G}}\big(\liminf_{j\to\infty} u_j\big) = \mathbb{E}^{\mathcal{G}}\big(\sup_k\inf_{j\geqslant k} u_j\big) = \sup_k \mathbb{E}^{\mathcal{G}}\big(\inf_{j\geqslant k} u_j\big) \leqslant \liminf_{m\to\infty}\mathbb{E}^{\mathcal{G}} u_m.$$

The Corollary is proved.

Proof of Corollary 23.12: Since $|u_j|\leqslant w$ we conclude that $|u| = \lim_j |u_j| \leqslant w$ and that $2w - |u-u_j| \geqslant 0$. Applying the conditional Fatou lemma 23.11 we find

$$\begin{aligned}\mathbb{E}^{\mathcal{G}}(2w) &= \mathbb{E}^{\mathcal{G}}\big(\liminf_j 2w - |u-u_j|\big)\\ &\leqslant \liminf_j \mathbb{E}^{\mathcal{G}}\big(2w - |u-u_j|\big)\\ &= \mathbb{E}^{\mathcal{G}}(2w) - \limsup_j \mathbb{E}^{\mathcal{G}}(|u-u_j|)\end{aligned}$$

which shows that

$$\limsup_j \mathbb{E}^{\mathcal{G}}(|u-u_j|) = 0 \implies \lim_j \mathbb{E}^{\mathcal{G}}(|u-u_j|) = 0.$$

Since, however,

$$\big|\mathbb{E}^{\mathcal{G}} u_j - \mathbb{E}^{\mathcal{G}} u\big| = \big|\mathbb{E}^{\mathcal{G}}(u_j - u)\big| \leqslant \mathbb{E}^{\mathcal{G}}|u_j - u| \xrightarrow{j\to\infty} 0$$

the claim follows.

Problem 23.8 (i) $\implies$ (ii): Let $A\in\mathcal{A}_\infty$ be such that $\mu(A)<\infty$. Then, by Hölder's inequality with $1/p+1/q=1$,

$$\left|\int_A u_j\,d\mu - \int_A u\,d\mu\right| \leqslant \int_A |u_j - u|\,d\mu \leqslant \|u_j - u\|_p\,\mu(A)^{1/q} \xrightarrow{j\to\infty} 0.$$

Thus, if $u_\infty := \mathbb{E}^{\mathcal{A}_\infty} u$, we find by the martingale property for all $k>j$ and $A\in\mathcal{A}_j$ such that $\mu(A)<\infty$

$$\int_A u_j\,d\mu = \int_A u_k\,d\mu = \lim_{k\to\infty}\int_A u_k\,d\mu = \int_A u\,d\mu = \int_A u_\infty\,d\mu,$$

and since we are in a σ-finite setting, we can apply Theorem 23.9(i) and find that $u_j = \mathbb{E}^{\mathcal{A}_j} u_\infty$.

(ii) $\implies$ (iii): Assume first that $u_\infty\in L^1\cap L^p$. Then $u_j = \mathbb{E}^{\mathcal{A}_j} u_\infty \in L^1\cap L^p$ and Theorem 23.15(i) shows that $u_j \xrightarrow{j\to\infty} u_\infty$ both in L^1 and a.e. In particular, we get

$$\langle u_\infty - u_j, \phi\rangle \leqslant \|u_\infty - u_j\|_1\|\phi\|_\infty \to 0 \quad \forall\,\phi\in L^\infty.$$

In the general case where $u_\infty\in L^p(\mathcal{A}_\infty)$ we find for every $\epsilon>0$ an element $u_\infty^\epsilon\in L^1(\mathcal{A}_\infty)\cap L^p(\mathcal{A}_\infty)$ such that

$$\|u_\infty - u_\infty^\epsilon\|_p \leqslant \epsilon$$

(indeed, since we are working in a σ-finite filtered measure space, there is an exhaustion $A_k \uparrow X$ such that $A_k \in \mathcal{A}_\infty$ and for large enough $k = k_\epsilon$ the function $u^\epsilon_\infty := u_\infty \mathbb{1}_{A_k}$ will to the job). Similarly, we can approximate any fixed $\phi \in L^q$ by $\phi^\epsilon \in L^q \cap L^1$ such that $\|\phi - \phi^\epsilon\|_q \leqslant \epsilon$.

Now we set $u^\epsilon_j := \mathbb{E}^{\mathcal{A}_j} u^\epsilon_\infty$ and observe that

$$\|u_j - u^\epsilon_j\|_p = \| \mathbb{E}^{\mathcal{A}_j} u_\infty - \mathbb{E}^{\mathcal{A}_j} u^\epsilon_\infty\|_p \leqslant \|u_\infty - u^\epsilon_\infty\|_p \leqslant \epsilon.$$

Thus, for any $\phi \in L^q$,

$$\begin{aligned}
&\langle u_j - u_\infty, \phi\rangle \\
&= \langle u_j - u^\epsilon_j - u_\infty + u^\epsilon_\infty, \phi\rangle + \langle u^\epsilon_j - u^\epsilon_\infty, \phi\rangle \\
&= \langle u_j - u^\epsilon_j - u_\infty + u^\epsilon_\infty, \phi\rangle + \langle u^\epsilon_j - u^\epsilon_\infty, \phi - \phi^\epsilon\rangle + \langle u^\epsilon_j - u^\epsilon_\infty, \phi^\epsilon\rangle \\
&\leqslant \big(\|u_j - u^\epsilon_j\|_p + \|u_\infty - u^\epsilon_\infty\|_p\big)\|\phi\|_q \\
&\qquad + \|u^\epsilon_j - u^\epsilon_\infty\|_p\|\phi - \phi^\epsilon\|_q + \langle u^\epsilon_j - u^\epsilon_\infty, \phi^\epsilon\rangle \\
&\leqslant 2\epsilon\,\|\phi\|_q + \epsilon \underbrace{\|u^\epsilon_j - u^\epsilon_\infty\|_p}_{\leqslant 2\|u^\epsilon_\infty\|_p \leqslant 2(\epsilon + \|u_\infty\|_p)} + \underbrace{\langle u^\epsilon_j - u^\epsilon_\infty, \phi^\epsilon\rangle}_{\xrightarrow{j\to\infty} 0} \qquad\qquad \leqslant \text{const.}\epsilon
\end{aligned}$$

for sufficiently large j's, and the claim follows.

(iii) $\Longrightarrow$ (ii): Let $u_{n(j)}$ be a subsequence converging weakly to some $u \in L^p$, i.e.,

$$\lim_k \langle u_{n(k)} - u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q.$$

Then, in particular,

$$\lim_k \langle u_{n(k)} - u, \mathbb{E}^{\mathcal{A}_n} \phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}$$

or

$$\lim_k \langle \mathbb{E}^{\mathcal{A}_n} u_{n(k)} - \mathbb{E}^{\mathcal{A}_n} u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}.$$

Since u_j is a martingale, we find that $\mathbb{E}^{\mathcal{A}_n} u_{n(k)}$ if $n < n(k)$, i.e.,

$$\langle u_n - \mathbb{E}^{\mathcal{A}_n} u, \phi\rangle = 0 \qquad \forall\, \phi \in L^q,\ n \in \mathbb{N}.$$

and we conclude that $u_n = \mathbb{E}^{\mathcal{A}_n} u$. Because of the tower property we can always replace u by $u_\infty := \mathbb{E}^{\mathcal{A}_\infty} u$:

$$u_n = \mathbb{E}^{\mathcal{A}_n} u = \mathbb{E}^{\mathcal{A}_n} \mathbb{E}^{\mathcal{A}_\infty} u = \mathbb{E}^{\mathcal{A}_n} u_\infty$$

and the claim follows.

(ii) $\Longrightarrow$ (i): We show that we can take $u = u_\infty$. First, if $u_\infty \in L^1 \cap L^\infty$ we find by the closability of martingales, Theorem 23.15(i), that

$$\lim_j \|u_j - u\|_1 = 0.$$

Moreover, using that $|a-b|^r \leqslant (|a|+|b|)^r \leqslant 2^r(|a|^r+|b|^r)$, we find

$$\begin{aligned}\|u_j-u\|_p^p &= \int |u_j-u|^p\,d\mu \\ &= \int |u_j-u|\cdot|u_j-u|^{p-1}\,d\mu \\ &\leqslant 2^{p-1}(\|u_j\|_\infty^{p-1}+\|u\|_\infty^{p-1})\int |u_j-u|\,d\mu \\ &\leqslant 2^p\|u\|_\infty^{p-1}\cdot\|u_j-u\|_1 \\ &\xrightarrow{j\to\infty} 0\end{aligned}$$

where we used that

$$\|u_j\|_\infty = \|\mathbb{E}_j^{\mathcal{A}} u\|_\infty \leqslant \mathbb{E}_j^{\mathcal{A}}\big(\|u\|_\infty\big) \leqslant \|u\|_\infty.$$

Now for the general case where $u_\infty \in L^p$. Since we are in a σ-finite setting, we can set $u^\epsilon := (u\cdot\mathbb{1}_{A_j})\wedge j$, $j=j(\epsilon)$ sufficiently large and $A_j \to X$ an exhausting sequence of sets from $\mathcal{A}_\infty$, and can guarantee that

$$\|u-u^\epsilon\|_p \leqslant \epsilon.$$

At the same time, we get for $u_j^\epsilon := \mathbb{E}^{\mathcal{A}_j} u^\epsilon \in L^1\cap L^\infty$ that

$$\|u_j-u_j^\epsilon\|_p = \|\mathbb{E}^{\mathcal{A}_j} u - \mathbb{E}^{\mathcal{A}_j} u^\epsilon\|_p \leqslant \|u-u^\epsilon\|_p \leqslant \epsilon.$$

Thus, by the consideration for the special case where $u^\epsilon \in L^1\cap L^\infty$,

$$\begin{aligned}\|u_j-u\|_p &\leqslant \|u_j-u_j^\epsilon\|_p + \|u_j^\epsilon-u^\epsilon\|_p + \|u_\epsilon-u\|_p \\ &\leqslant \epsilon + \|u_j^\epsilon-u^\epsilon\|_p+\epsilon \\ &\xrightarrow{j\to\infty} 2\epsilon \xrightarrow{\epsilon\to 0} 0.\end{aligned}$$

Problem 23.9 Obviously,

$$m_k = m_{k-1} + (u_k - E^{\mathcal{A}_{k-1}}u_k).$$

Since $m_1 = u_1 \in L^1\mathcal{A}_1$, this shows, by induction, that $m_k \in L^1(\mathcal{A}_k)$. Applying $E^{\mathcal{A}_{k-1}}$ to both sides of the displayed equality yields

$$\begin{aligned}E^{\mathcal{A}_{k-1}}m_k &= E^{\mathcal{A}_{k-1}}m_{k-1} + E^{\mathcal{A}_{k-1}}(u_k - E^{\mathcal{A}_{k-1}}u_k) \\ &= m_{k-1} + E^{\mathcal{A}_{k-1}}u_k - E^{\mathcal{A}_{k-1}}u_k \\ &= m_{k-1}\end{aligned}$$

which shows that m_k is indeed a martingale.

Problem 23.10 Problem 23.9 shows that s_k is a martingale, so that s_k^2 is a sub-martingale (use Jensen's inequality for conditional expectations). Now

$$\int s_k^2\,d\mu = \sum_j \int u_k^2\,d\mu + 2\sum_{j<k}\int u_j u_k\,d\mu$$

and if $j < k$

$$\int u_j u_k \, d\mu = \int E^{\mathcal{A}_j}(u_j u_k) \, d\mu = \int u_j \underbrace{E^{\mathcal{A}_j}(u_k)}_{=0} \, d\mu = 0.$$

Problem 23.11 **Mind the misprint: $s_j = u_j$ so that $s_0 = u_0 = 0$.** Problem 23.9 shows that m_j is a martingale.

Since $a_1 = E^{\mathcal{A}_0} u_1 - u_0 = E^{\{\emptyset, X\}} u_1 = \int u_1 \, d\mu$ is constant, i.e., $\mathcal{A}_0$-measurable, the recursion formula

$$a_{j+1} = a_j + E^{\mathcal{A}_j} u_{j+1} - u_j$$

implies that a_{j+1} is $\mathcal{A}_j$-measurable.

Since u_j is a submartingale, we get

$$E^{\mathcal{A}_j} u_{j+1} \geqslant u_j \implies a_{j+1} - a_j \geqslant 0$$

i.e., the sequence a_j increases.

Finally, if $m_j + a_j = u_j = \tilde{m}_j + \tilde{a}_j$ are two such decompositions we find that $m_j - \tilde{m}_j = a_j - \tilde{a}_j$ is $\mathcal{A}_{j-1}$ measurable. Using the martingale property we find

$$m_j - \tilde{m}_j = E^{\mathcal{A}_{j-1}}(m_j - \tilde{m}_j) \overset{\text{Martingale}}{=} m_{j-1} - \tilde{m}_{j-1}$$

and applying this recursively for $j = 1, 2, 3, \ldots$ yields

$$m_1 - \tilde{m}_1 = 0, \quad m_2 - \tilde{m}_2 = 0, \quad m_3 - \tilde{m}_3 = 0, \ldots$$

so that $m_j = \tilde{m}_j$ and, consequently, $a_j = \tilde{a}_j$.

Problem 23.12 Assume that $M_k = E^{\mathcal{A}_k} M$. Then we know from Theorem 23.15 that $\tilde{M} = \lim_k M_k$ exists a.e. and in L^1. Moreover, $\int M_k \, dP = 1$ so that $\tilde{M}$ cannot be trivial. On the other hand,

$$P(\tilde{M} > 0) \leqslant P(M_k > 0) = P(X_j > 0 \quad \forall j = 1, 2, \ldots, k) = 2^{-k} \xrightarrow{k \to \infty} 0$$

which yields a contradiction.

24 Orthonormal systems and their convergence behaviour
Solutions to Problems 24.1–24.11

Problem 24.1 Since $J_k^{(\alpha,\beta)}$ is a polynomial of degree k, it is enough to show that $J_k^{(\alpha,\beta)}$ is orthogonal in $L^2(I, \rho(x)\,dx)$ to any polynomial $p(x)$ of degree $j < k$. We write ∂^k for $\frac{d^k}{dx^k}$ and $u(x) = (x-1)^{k+\alpha}(x+1)^{k+\beta}$. Then we get by repeatedly integrating by parts

$$\begin{aligned}
&\int_{-1}^{1} J_k^{(\alpha,\beta)}(x)p(x)(x-1)^{\alpha}(x+1)^{\beta}\,dx \\
&= \frac{(-1)^k}{k!\,2^k}\int_{-1}^{1} p(x)\partial^k u(x)\,dx \\
&= \Big[p(x)\cdot\partial^{k-1}u(x) - \partial^1 p(x)\cdot\partial^{k-2}u(x) + \cdots + (-1)^{k-1}\partial^{k-1}p(x)\cdot u(x)\Big]_{-1}^{1} \\
&\qquad + (-1)^k\int_{-1}^{1} u(x)\partial^k p(x)\,dx.
\end{aligned}$$

Obviously, $\partial^\ell u(-1) = \partial^\ell u(1) = 0$ for all $0 \leqslant \ell \leqslant k-1$ and $\partial^k p \equiv 0$ since p is a polynomial of degree $j < k$.

Problem 24.2 It is pretty obvious how to go about this problem. The calculations themselves are quite tedious and therefore omitted.

Problem 24.3

Theorem 24.6': *The polynomials are dense in $C[a,b]$ with respect to uniform convergence.*

Proof 1: mimic the proof of 24.6 with the obvious changes;

Proof 2: Let $f \in C[a,b]$. Then $\tilde f(y) := f(a+(b-a)y), y \in [0,1]$ satisfies $\tilde f \in C[0,1]$ and, because of Theorem 24.6, there is a sequence of polynomials $\tilde p_n$ such that

$$\lim_{n\to\infty}\sup_{y\in[0,1]}|\tilde f(y) - \tilde p_n(y)| = 0.$$

Define $p_n(x) := \tilde p_n\big(\frac{x-a}{b-a}\big), x \in [a,b]$. Clearly p_n is a polynomial and we have

$$\sup_{x\in[a,b]}|p_n(x) - f(x)| = \sup_{y\in[0,1]}|\tilde p_n(y) - \tilde f(y)|.$$

Corollary 24.8': *The monomials are complete in $L^1([a,b], dt)$.*

Proof 1: mimic the proof of 24.8 with the obvious changes;

Proof 2: assume that for all $j \in \mathbb{N}_0$ we have

$$\int_a^b u(x)x^j\,dx = 0.$$

Since

$$\begin{aligned}\int_0^1 u((b-a)t+a)t^j\,dx &= \int_a^b u(x)\left[\frac{x-a}{b-a}\right]^j dx\\ &= \sum_{k=0}^{j} c_k \int_a^b u(x)x^k\,dx\\ &= 0\end{aligned}$$

we get from Corollary 24.8 that

$$u((b-a)t+a) = 0 \qquad \text{Lebesgue almost everywhere on } [0,1]$$

and since the map $[0,1] \ni t \mapsto x = (b-a)t + a \in [a,b]$ is continuous, bijective and with a continuous inverse, we also get

$$u(x) = 0 \qquad \text{Lebesgue almost everywhere on } [a,b].$$

Problem 24.4 Observe that

$$\begin{aligned}\operatorname{Re}\left(e^{i(x-y)} - e^{i(x+y)}\right) &= \operatorname{Re}\left[e^{ix}\left(e^{-iy} - e^{iy}\right)\right]\\ &= \operatorname{Re}\left[-2ie^{ix}\sin y\right]\\ &= 2\sin x \sin y,\end{aligned}$$

and that

$$\begin{aligned}\operatorname{Re}\left(e^{i(x+y)} + e^{i(x-y)}\right) &= \operatorname{Re}\left[e^{ix}\left(e^{iy} + e^{-iy}\right)\right]\\ &= \operatorname{Re}\left[2e^{ix}\cos y\right]\\ &= 2\cos x \cos y.\end{aligned}$$

Moreover, we see that for $N \in \mathbb{N}_0$

$$\int_{-\pi}^{\pi} e^{iNx}\,dx = \begin{cases}\frac{e^{iNx}}{iN}\Big|_{-\pi}^{\pi} = 0, & \text{if } N \neq 0;\\ 2\pi, & \text{if } N = 0.\end{cases}$$

Thus, if $k \neq \ell$

$$\int_{-\pi}^{\pi} 2\cos kx \cos \ell x\,dx = \operatorname{Re}\left(\int_{-\pi}^{\pi} e^{i(k+\ell)x}\,dx + \int_{-\pi}^{\pi} e^{i(k+\ell)x}\,dx\right) = 0$$

and if $k = \ell \geqslant 1$

$$\int_{-\pi}^{\pi} 2\cos kx \cos kx\,dx = \operatorname{Re}\left(\int_{-\pi}^{\pi} e^{2ikx}\,dx + \int_{-\pi}^{\pi} 1\,dx\right) = 2\pi$$

and if $k = \ell = 0$,

$$\int_{-\pi}^{\pi} 2\cos kx \cos kx \, dx = \int 2 \, dx = 4\pi.$$

The proof for the pure sines integral is similar while for the mixed sine-cosine integrals the integrand

$$x \mapsto \cos kx \sin \ell x$$

is always an odd function, the integral over the symmetric (w.r.t. the origin) interval $(-\pi, \pi)$ is always zero.

Problem 24.5 (i) We have

$$\begin{aligned} 2^k \cos^k(x) &= 2^k \Big(\frac{e^{ix} + e^{-ix}}{2} \Big)^k \\ &= \Big(\frac{e^{ix} + e^{-ix}}{2} \Big)^k \\ &= \sum_{j=0}^{k} \binom{k}{j} e^{ijx} e^{-i(k-j)x} \\ &= \sum_{j=0}^{k} \binom{k}{j} e^{i(2j-k)x} \end{aligned}$$

Adding the first and last terms, second and penultimate terms, term no. j and $k-j$, etc. under the sum gives, since the binomial coefficients satisfy $\binom{k}{j} = \binom{k}{k-j}$,

– if $k = 2n$ is even

$$\begin{aligned} 2^{2n} \cos^{2n}(x) &= \sum_{j=0}^{n-1} \binom{2n}{j} (e^{i(2j-2n)x} + e^{i(2n-2j)x}) + \binom{2n}{n} \\ &= \sum_{j=0}^{n} \binom{2n}{j} 2\cos(2j-2n) + \binom{2n}{n} \end{aligned}$$

– if $k = 2n-1$ is odd

$$\begin{aligned} 2^{2n-1} \cos^{2n-1}(x) &= \sum_{j=0}^{n-1} \binom{2n-1}{j} (e^{i(2j-2n+1)x} + e^{i(2n-2j-1)x}) \\ &= \sum_{j=0}^{n-1} \binom{2n-1}{j} 2\cos(2n-2j-1)x. \end{aligned}$$

In a similar way we compute $\sin^k x$:

$$\begin{aligned} 2^k \sin^k(x) &= 2^k \Big(\frac{e^{ix} - e^{-ix}}{2i} \Big)^k \\ &= i^{-k} \Big(\frac{e^{ix} - e^{-ix}}{2} \Big)^k \\ &= i^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{ijx} e^{-i(k-j)x} \\ &= i^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} e^{i(2j-k)x}. \end{aligned}$$

Adding the first and last terms, second and penultimate terms, term no. j and $k-j$, etc. under the sum gives, since the binomial coefficients satisfy $\binom{k}{j} = \binom{k}{k-j}$,

– if $k = 2n$ is even

$$\begin{aligned}
&2^{2n}\sin^{2n}(x)\\
&= (-1)^n \sum_{j=0}^{n-1}\binom{2n}{j}\Big((-1)^{2n-j}e^{i(2j-2n)x} + (-1)^j e^{i(2n-2j)}\Big) + \binom{2n}{n}\\
&= \sum_{j=0}^{n-1}\binom{2n}{j}(-1)^{n-j}\Big(e^{i(2j-2n)x} + e^{i(2n-2j)}\Big) + \binom{2n}{n}\\
&= \sum_{j=0}^{n-1}\binom{2n}{j}(-1)^{n-j}2\cos(2n-2j)x + \binom{2n}{n}
\end{aligned}$$

– if $k = 2n-1$ is odd

$$\begin{aligned}
&2^{2n-1}\sin^{2n-1}(x)\\
&= i(-1)^n \sum_{j=0}^{n-1}\binom{2n-1}{j}\Big((-1)^{2n-1-j}e^{i(2j-2n+1)x} + (-1)^{-j}e^{i(2n-2j-1)}\Big)\\
&= i\sum_{j=0}^{n-1}\binom{2n-1}{j}(-1)^{n-j}\Big(-e^{i(2j-2n+1)x} + e^{i(2n-2j-1)}\Big)\\
&= i\sum_{j=0}^{n-1}\binom{2n-1}{j}(-1)^{n-j}2i\sin(2n-2j+1)x\\
&= \sum_{j=0}^{n-1}\binom{2n-1}{j}(-1)^{n-j-1}2\sin(2n-2j+1)x.
\end{aligned}$$

(ii) We have

$$\cos kx + i\sin kx = e^{ikx} = \big(e^{ix}\big)^k = \big(\cos x + i\sin x\big)^k$$

and we find, using the binomial formula,

$$\cos kx + i\sin kx = \sum_{j=0}^{k}\binom{k}{j}\cos^j x\cdot i^{k-j}\sin^{k-j}x$$

and the claim follows by separating real and imaginary parts.

(iii) Since a trigonometric polynomial is of the form

$$T_n(x) = a_0 + \sum_{k=1}^{n}\big(a_k\cos kx + b_k\sin kx\big)$$

it is a matter of double summation and part (ii) to see that $T_n(x)$ can be written like $U_n(x)$.

Conversely, part (i) enables us to rewrite any expression of the form $U_n(x)$ as $T_n(x)$.

Problem 24.6 By definition,

$$D_N(x) = \frac{1}{2} + \sum_{j=1}^{N}\cos jx.$$

Multiplying both sides by $\sin\frac{x}{2}$ and using the formula

$$\cos ax\sin bx = \frac{1}{2}\Big(\sin\frac{(a+b)x}{2} - \sin\frac{(a-b)x}{2}\Big)$$

where $j = (a+b)/2$ and $1/2 = (a-b)/2$, i.e. $a = (2j+1)/2$ and $b = (2j-1)/2$ we arrive at

$$D_N(x)\sin\tfrac{x}{2} = \tfrac{1}{2}\sin\tfrac{x}{2} + \tfrac{1}{2}\sum_{j=1}^{N}\Big(\sin\tfrac{(2j+1)x}{2} - \sin\tfrac{(2j-1)x}{2}\Big) = \sin\tfrac{(2N+1)x}{2}.$$

Problem 24.7 We have

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right).$$

Indeed, let us calculate the Fourier coefficients 24.8. First,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\sin kx\,dx = 0, \qquad k\in\mathbb{N},$$

since the integrand is an odd function. So no sines appear in the Fourier series expansion. Further, using the symmetry properties of the sine function

$$\begin{aligned}
a_0/2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin x|\,dx\\
&= \frac{1}{\pi}\int_{0}^{\pi}|\sin x|\,dx\\
&= \frac{1}{\pi}(-\cos x)\Big|_0^{\pi}\\
&= \frac{2}{\pi}
\end{aligned}$$

and using the elementary formula $2\sin a\cos b = \sin(a-b)+\sin(a+b)$ we get

$$\begin{aligned}
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cos jx\,dx\\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin x\cos jx\,dx\\
&= \frac{2}{\pi}\int_{0}^{\pi}\frac{1}{2}\big(\sin((j+1)x) - \sin((j-1)x)\big)\,dx\\
&= \frac{1}{\pi}\left[\frac{\cos((j-1)x)}{j-1} - \frac{\cos((j+1)x)}{j+1}\right]_0^{\pi}\\
&= \frac{1}{\pi}\left[\frac{\cos((j-1)\pi)}{j-1} - \frac{\cos((j+1)\pi)}{j+1} - \frac{1}{j-1} + \frac{1}{j+1}\right].
\end{aligned}$$

If j is odd, we get $a_j = 0$ and if j is even, we have

$$a_j = \frac{1}{\pi}\left[\frac{-1}{j-1} - \frac{-1}{j+1} - \frac{1}{j-1} + \frac{1}{j+1}\right] = -\frac{4}{\pi}\,\frac{1}{(j-1)(j+1)}.$$

This shows that we have only evenly indexed cosines in the Fourier series.

Problem 24.8 This is not as trivial as it looks in the first place! Since u is itself a Haar function, we have

$$s_N(u,x) = u(x) \qquad \forall N\in\mathbb{N}$$

(it is actually the first Haar function) so that s_N converges in any L^p-norm, $1\leqslant p<\infty$ to u.

The same applies to the *right tail* of the Haar wavelet expansion. The left tail, however, converges only for $1 < p < \infty$ in L^p. The reason is the calculation of step 5 in the proof of Theorem 24.20 which goes in the case $p = 1$:

$$\begin{aligned}\mathbb{E}^{\mathcal{A}^\Delta_{-M}} u &= 2^{-M} \int_{[-2^M,0)} u(x)\,dx\,\mathbb{1}_{[-2^M,0)} + 2^{-M} \int_{[0,2^M)} u(x)\,dx\,\mathbb{1}_{[0,2^M)}\\ &= 2^{-M}\,\mathbb{1}_{[0,2^M)},\end{aligned}$$

but this is not L^1-convergent to 0 as it would be required. For $p > 1$ all is fine, though....

Problem 24.9 Assume that u is uniformly continuous (C_c and C_∞-functions are!). Since

$$s_n(u;x) = \mathbb{E}^{\mathcal{A}^H_n} u(x)$$

is the projection onto the sets in $\mathcal{A}^H_n$, see e.g. step 2 in the proof of Theorem 24.17, we have

$$s_n(u;x) = \frac{1}{\lambda(I)} \int_I u(y)\,dx\,\mathbb{1}_I(x)$$

where I is an dyadic interval from the generator of $\mathcal{A}^H_n$ as in step 2 of the proof of Theorem 24.17. Thus, if x is from I we get

$$\begin{aligned}|s_n(u;x) - u(x)| &= \left|\frac{1}{\lambda(I)} \int_I (u(y) - u(x))\,dx\right|\\ &\leqslant \frac{1}{\lambda(I)} \int_I |u(y) - u(x)|\,dx\\ &\leqslant \frac{1}{\lambda(I)} \int_I \epsilon\,dx\\ &= \epsilon\end{aligned}$$

if $\lambda(I) < \delta$ for small enough $\delta > 0$. This follows from uniform continuity: for given $\epsilon > 0$ there is some $\delta > 0$ such that for $x, y \in I$ (this entails $|x - y| \leqslant \delta$!) we have $|u(x) - u(y)| \leqslant \epsilon$.

The above calculation holds uniformly for all x and we are done.

Problem 24.10 The calculation for the right tail is more or less the same as in Problem 24.9. Only the left tail differs. Here we argue as in step 5 of the proof of Theorem 24.20: if $u \in C_c(\mathbb{R})$ we can assume that $\operatorname{supp} u \subset [-R, R]$ and we see

$$\begin{aligned}\mathbb{E}^{\mathcal{A}^\Delta_{-M}} u(x) &= 2^{-M} \int_{[-R,0]} u(x)\,dx\,\mathbb{1}_{[-2^M,0)} + 2^{-M} \int_{[0,R]} u(x)\,dx\,\mathbb{1}_{[0,2^M)}\\ &\leqslant 2^{-M} R\,\|u\|_\infty\,\mathbb{1}_{[-2^M,0)} + 2^{-M} R\,\|u\|_\infty\,\mathbb{1}_{[0,2^M)}\\ &= 2^{-M} R\,\|u\|_\infty\,\mathbb{1}_{[-2^M,2^M)}\\ &\leqslant 2^{-M} R\,\|u\|_\infty \xrightarrow[\text{uniformly for all } x]{M\to\infty} 0\end{aligned}$$

If $u \in C_\infty$ we can use the fact that C_c is dense in C_∞, i.e. we can find for every $\epsilon > 0$ functions $v = v_\epsilon \in C_c$ and $w = w_\epsilon \in C_\infty$ such that

$$u = v + w \quad \text{and} \quad \|w\|_\infty \leqslant \epsilon.$$

Then

$$\begin{aligned}\left|\mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} u(x)\right| &\leqslant \left|\mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} v(x)\right| + \left|\mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} w(x)\right| \\ &\leqslant \left|\mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} v(x)\right| + \mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} \|w\|_\infty \\ &\leqslant \left|\mathbb{E}^{\mathcal{A}^{\Delta}_{-M}} v(x)\right| + \epsilon\end{aligned}$$

and, by the first calculation for C_c-functions, the right-hand side converges, since $v \in C_c$, to $0 + \epsilon$ uniformly for all x, and letting $\epsilon \to 0$ we conclude the proof.

Problem 24.11 See the picture at the end of this solution.

Since the function $u(x) := \mathbb{1}_{[0,1/3)}(x)$ is piecewise constant, and since for each Haar function $\int \chi_{j,k}\,dx = 0$ unless $j = k = 1$, we see that only a single Haar function contributes to the value of $s_N(u; \frac{1}{3})$, namely where $\frac{1}{3} \in \operatorname{supp} \chi_{j,n}$.

The idea of the proof is now pretty clear: take values N where $x = \frac{1}{3}$ is in the left 'half' of $\chi_{k,N}$, i.e. where $\chi_{k,N}(\frac{1}{3}) = 1$ and values M such that $x = \frac{1}{3}$ is in the opposite, negative 'half' of $\chi_{\ell,M}$, i.e. where $\chi_{\ell,M}(\frac{1}{3}) = -1$. Of course, k, ℓ depend on x, N and M respectively. One should expect that the partial sums for these different positions lead to different limits, hence different upper and lower limits.

The problem is to pick N's and M's. We begin with the simple observation that the dyadic (i.e. base-2) representation of 1/3 is the periodic, infinite dyadic fraction

$$\frac{1}{3} = 0.01010101\cdots = \sum_{k=1}^{\infty} \frac{1}{2^{2k}}$$

and that the finite fractions

$$d_n := 0.\underbrace{0101\cdots01}_{2n} = \sum_{k=1}^{n} \frac{1}{2^{2k}}$$

approximate 1/3 from the left in such a way that

$$\frac{1}{3} - d_n = \sum_{k=n+1}^{\infty} \frac{1}{2^{2k}} < \sum_{\ell=2n+2}^{\infty} \frac{1}{2^{\ell}} = \frac{1}{2^{2n+2}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{2n+1}}$$

Now consider those Haar functions whose support consists of intervals of the length 2^{-2n}, i.e. the $\chi_{j,2n}$'s and agree that $j = j(1/3, n)$ is the one value where $\frac{1}{3} \in \operatorname{supp} \chi_{j,2n}$. By construction $\operatorname{supp} \chi_{j,2n} = [d_n, d_n + 1/2^{2n}]$ and we get for the Haar-Fourier partial sum

$$\begin{aligned}s_{2n}(u, \tfrac{1}{3}) - \frac{1}{3} &= \int_{d_n}^{1/3} 2^n dx \cdot \chi_{j,2n}(\tfrac{1}{3}) \\ &= 2^{2n}\left(\tfrac{1}{3} - d_n\right) \\ &= 4^n \sum_{k=n+1}^{\infty} \frac{1}{2^{2k}} \\ &= 4^n \sum_{k=n+1}^{\infty} \frac{1}{4^k}\end{aligned}$$

$$= 4^n 4^{-n-1} \frac{1}{1-\frac{1}{4}} = \frac{1}{3}.$$

The shift by $-1/3$ comes from the starting 'atypical' Haar function $\chi_{0,0}$ since $\langle u, \chi_{0,0}\rangle = \int_0^{1/3} dx = \frac{1}{3}$.

Using the next smaller Haar functions with support of length 2^{-2n-1}, i.e. the $\chi_{k,2n+1}$'s, we see that with j as above $\chi_{2j-1,2n+1}(\frac{1}{3}) = -1$ (since twice as many Haar functions appear in the run-up to d_n) and that

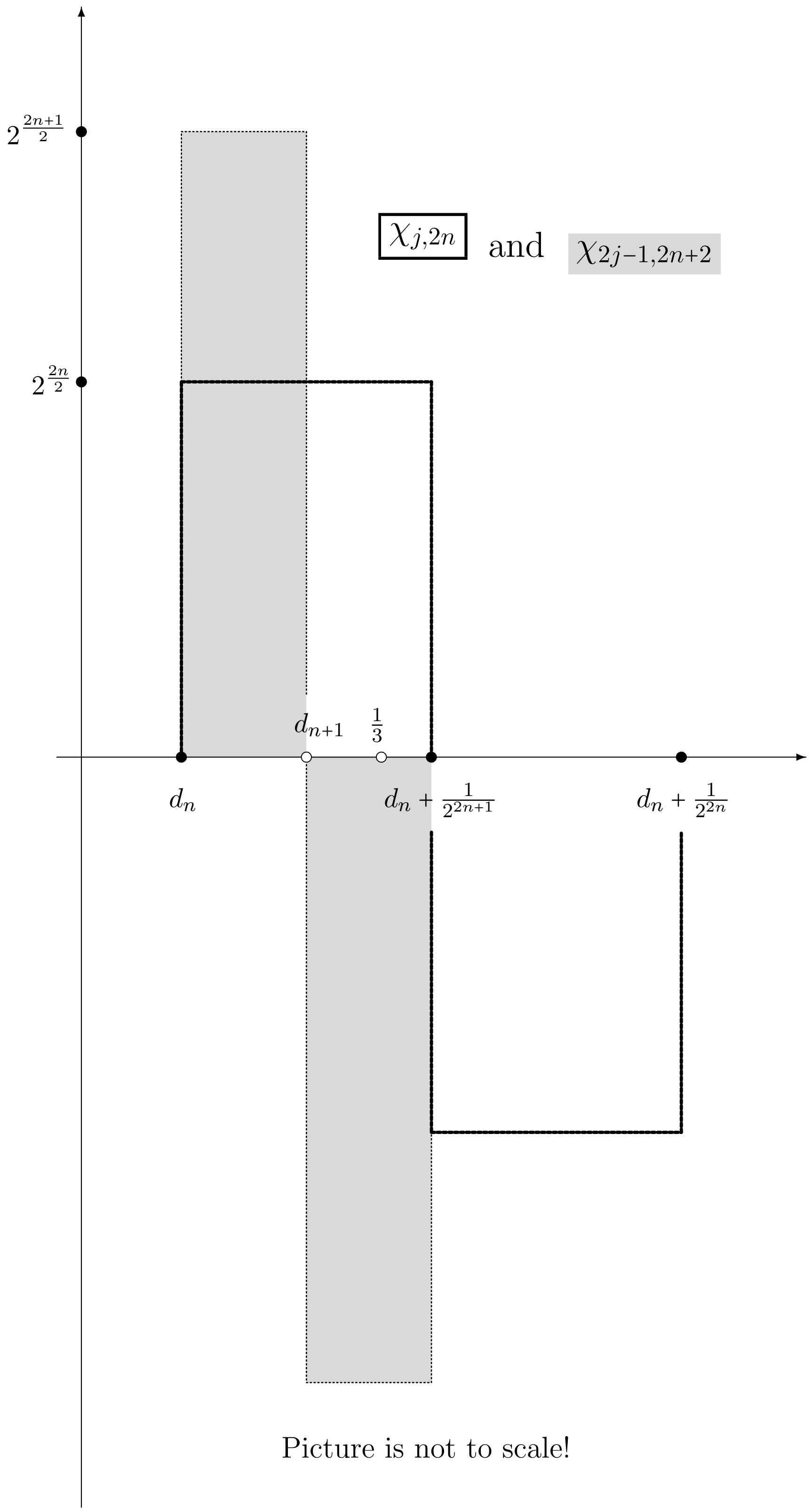
$$\begin{aligned}
& s_{2n+1}(u, \tfrac{1}{3}) - \frac{1}{3} \\
&= \left[\int_{d_n}^{d_n+1/2^{2n+2}} 2^{n+1} dx - \int_{d_n+1/2^{2n+2}}^{1/3} 2^{n+1} dx\right] \cdot \chi_{2j-1,2n+1}(\tfrac{1}{3}) \\
&= \left[d_n + \frac{1}{2^{2n+2}} - d_n - \frac{1}{3} + d_n + \frac{1}{2^{2n+2}}\right] 2^{n+1} \cdot (-2^{n+1}) \\
&= \left[d_n - \frac{1}{3} + \frac{2}{2^{2n+2}}\right] \cdot (-2^{2n+2}) \\
&= 4 \cdot 2^{2n}\left(\frac{1}{3} - d_n\right) - 2 \\
&= 4 \cdot \frac{1}{3} - 2 && \text{(using the result above)} \\
&= -\frac{2}{3}
\end{aligned}$$

This shows that

$$s_{2n}(u; \tfrac{1}{3}) = \frac{2}{3} > -\frac{1}{3} = s_{2n+1}(u, \tfrac{1}{3})$$

and the claim follows since because of the above inequality,

$$\liminf_N s_N(u; \tfrac{1}{3}) \leqslant -\frac{1}{3} \leqslant \frac{2}{3} \leqslant \limsup_N s_N(u; \tfrac{1}{3}).$$



Picture is not to scale!